Homework #5

1. Define $f : \mathbb{T} \to \mathbb{R}$ by $f(x) = x^2$ for $-\pi \le x \le \pi$.
   (a) Compute the Fourier coefficients of f.
   (b) Use Parseval's identity to deduce that
   $$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

2. Suppose that $\{\phi_n\}$ is an approximate identity on $\mathbb{T}$ and $f \in L^1(\mathbb{T})$. Prove that
   $$\|\phi_n * f\|_1 \le \|f\|_1$$
   and
   $$\|\phi_n * f - f\|_1 \longrightarrow 0$$
   as $n \to \infty$.

3. Consider the differential equation
   $$-u'' + u = f.$$
   If $f \in L^2(\mathbb{T})$, use Fourier series to show that there is a unique solution $u \in H^2(\mathbb{T})$.

4. Let $T, S \in L^2(\mathbb{T})$ be the triangular and square wave, respectively, defined by
   $$T(x) = |x| \quad \text{if } |x| \le \pi, \qquad S(x) = \begin{cases} 1 & \text{if } 0 < x < \pi, \\ -1 & \text{if } -\pi < x < 0. \end{cases}$$
   (a) Compute the Fourier series of T and S.
   (b) Show that $T \in H^1(\mathbb{T})$ and $T' = S$.
   (c) Show that $S \notin H^1(\mathbb{T})$.