Homework #4

1. Suppose $\{u_n\}$ is an orthonormal sequence in a Hilbert space $\mathcal{H}$. For any $x \in \mathcal{H}$, prove that

$$\lim_{n \to \infty} (x, u_n) = 0.$$

2. Prove that if $\mathcal{M}$ is a dense linear subspace of a separable Hilbert space $\mathcal{H}$, then $\mathcal{H}$ has an orthonormal basis consisting of elements of $\mathcal{M}$. Does the same result hold for arbitrary dense subsets of $\mathcal{H}$?

3. Define the **Legendre polynomials** P_n by

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n.$$

   (a) Show that the Legendre polynomials are orthogonal in $L^2([-1, 1], m)$, and that they are obtained by Gram–Schmidt orthogonalization (without proper normalization) of the monomials.

   (b) Show that

$$\|P_n\|^2 = \int_{-1}^{1} P_n(x)^2 dx = \frac{2}{2n+1}.$$

   (c) Define $Q_n(x) = P_n(x)/\|P_n\|$. Show that the polynomials $\{Q_n : n \in \mathbb{N}\}$ form an orthonormal basis of $L^2([-1, 1], m)$.

Bonus: Prove that any two orthonormal bases of the Hilbert space $\mathcal{H}$ have the same cardinality.