Homework #2

1. Let ν be a signed measure on $(X, \mathcal{F})$. For $E \in \mathcal{F}$, define

$$\lambda(E) = \sup \sum_j |\nu(E_j)|,$$

where the supremum is over all finite partitions $\{E_j\}$ of E. Does it follow that $\lambda = |\nu|$? If yes, prove the statement. Otherwise provide a counterexample.

2. If $f \in L^1(\mathbb{R}^n)$ and $f \neq 0$, show that there exists $C, R > 0$ such that $Hf(x) \geq C\|x\|^{-n}$ for $\|x\| > R$.

3. For a signed Borel measure μ on $\mathbb{R}^n$, we define the **maximal function** $M\mu$ by

$$(M\mu)(x) = \sup_{r>0} \frac{|\mu|(B(x,r))}{m(B(x,r))}.$$

Prove that there exists a constant $C > 0$ such that for every finite signed Borel measure μ on $\mathbb{R}^n$ and every $\alpha > 0$

$$m(\{x \in \mathbb{R}^n : (M\mu)(x) > \alpha\}) \leq \frac{C}{\alpha}|\mu|(\mathbb{R}^n).$$

4. If E is a Borel set in $\mathbb{R}^n$, the **density** $D_E(x)$ of E at the point x is defined as

$$D_E(x) = \lim_{r\to 0} \frac{m(E \cap B(x,r))}{m(B(r,x))},$$

whenever the limit exists.

   (a) Show that $D_E(x) = 1$ for a.e. $x \in E$ and $D_E(x) = 0$ for a.e. $x \notin E$.

   (b) Find an example of E and x such that $D_E(x) \in (0,1)$ or $D_E(x)$ does not exist.

5. If λ and μ are positive, mutually singular Borel measures on $\mathbb{R}^n$ and $\lambda + \mu$ is regular, show that λ and μ are also both regular.

6. (a) If $F : \mathbb{R} \to \mathbb{R}$ is increasing and bounded, show that $F \in BV$.

   (b) If $F, G \in BV$ and $a, b \in \mathbb{C}$, prove that $aF + bG \in BV$.