# Homework #1

1. If $\nu$ is a signed measure on $(X, \mathcal{F})$, prove that $E \in \mathcal{F}$ is null for $v$ if and only if $|\nu|(E) = 0$. In addition, for signed measures $\mu$ and $\nu$ on $(X, \mathcal{F})$, show that the following are equivalent
   (a) $\nu \perp \mu$
   (b) $|\nu| \perp \mu$
   (c) $\nu^+ \perp \mu$ and $\nu^- \perp \mu$
2. Suppose $\nu(E) = \int_E f d\mu$, where $\mu$ is a positive measure on $(X, \mathcal{F})$ and $f : X \to [-\infty, \infty]$ is a measurable function such that at least one of $\int f^+ d\mu$ or $\int f^- d\mu$ is finite. Describe the Hahn decomposition of $\nu$ as well as the positive, negative, and total variations of $\nu$ in terms of $f$ and $\mu$.
3. Let $\nu$ be a signed measure on $(X, \mathcal{F})$, and assume $\mu$ is a positive measure on $(X, \mathcal{F})$. Prove that the following are equivalent:
   (a) $\nu \ll \mu$,
   (b) $|\nu| \ll \mu$,
   (c) $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.
4. Let $\{\nu_n\}$ be a sequence of positive measures, and let $\mu$ be a positive measure on $(X, \mathcal{F})$.
   (a) If $\nu_n \perp \mu$ for all $n$, show that $\sum_{n=1}^{\infty} \nu_n \perp \mu$.
   (b) If $\nu_n \ll \mu$ for all $n$, prove that $\sum_{n=1}^{\infty} \nu_n \ll \mu$.
5. Let $\mu$ be a positive measure. A collection of functions $\{f_\alpha\}_{\alpha \in I}$ in $L^1(\mu)$ is called **uniformly integrable** if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

   $$\sup_{\alpha \in I} \left| \int_A f_\alpha \, d\mu \right| < \varepsilon$$

   whenever $\mu(A) < \delta$.
   (a) Prove that any finite subset of $L^1(\mu)$ is uniformly integrable.
   (b) If $\{f_n\}$ is a sequence in $L^1(\mu)$ which converges in $L^1$-metric to $f \in L^1(\mu)$, show that $\{f_n\}$ is uniformly integrable.
6. Prove the uniqueness portion of the Lebesgue–Radon–Nikodym theorem.
7. Let $X = [0, 1]$, $\mathcal{F}$ be the Borel $\sigma$-algebra on $X$, $m$ be Lebesgue measure, and $\mu$ be the counting measure on $\mathcal{F}$. Show that $m \ll \mu$ but $dm \neq f d\mu$ for any $f$. Does this contradict the Radon–Nikodym theorem? Explain.