Final Exam

INSTRUCTIONS:

1. Answer each question on a separate page. Turn in a page for each problem even if you cannot do the problem.
2. Label each answer sheet with the problem number.
3. Put your name in the upper right hand corner of each page.

PROBLEMS:

1. Let X be a normed linear space and Y a Banach space. If M is a dense linear subspace of X and $T : M \to Y$ is a bounded linear map, prove that there exists a unique bounded linear extension $\overline{T} : X \to Y$ such that $\overline{T}x = Tx$ for all $x \in M$ and $\|\overline{T}\| = \|T\|$. This result is known as the **Bounded Linear Transformation (BLT) theorem**.
2. Let $f \in \mathcal{S}(\mathbb{R})$ with the property that
$$\int_{\mathbb{R}} f(x)x^n dx = 0$$
for every $n = 0, 1, 2, \ldots$. Must f be identically zero? Prove or disprove.
3. Let μ be a Radon measure on the locally compact group G. If $f \in C_c(G)$, prove that the functions $x \mapsto \int L_x f d\mu$ and $x \mapsto \int R_x f d\mu$ are continuous.
4. Consider a linear functional $T(f) = f(1/2)$ defined on the space of polynomials on $[0, 1]$. Does T extend to a bounded linear functional on $L^2([0, 1])$? Prove or disprove.
5. Let G be a topological group and H a subgroup of G. Prove that
$$\overline{H} = \bigcap_U HU,$$
where the intersection is over all neighborhoods U of e.
6. If f is a real-valued function on $[0, 1]$ and
$$\gamma(t) = t + if(t),$$
the length of the graph of f is, by definition, the total variation of γ on $[0, 1]$. Show that this length is finite if and only if $f \in BV([0, 1])$. In addition, show that it is equal to
$$\int_0^1 \sqrt{1 + |f'(t)|^2} dt$$
if f is absolutely continuous.