Homework # 1

1. Find an example of a set X and two σ-algebras $\mathcal{A}_1, \mathcal{A}_2$ on X such that $\mathcal{A}_1 \cup \mathcal{A}_2$ is not a σ-algebra.
2. Prove that every σ-finite measure is semifinite.
3. Let μ be a finitely additive measure on the measurable space $(X, \mathcal{A})$. Assume μ satisfies the following property:

$$\text{if } A_1 \subseteq A_2 \subseteq \cdots \text{ are subsets of } \mathcal{A}, \text{ then } \mu(\cup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n).$$

   Show that μ is a measure on $(X, \mathcal{A})$.
4. Suppose $(X, \mathcal{A}, \mu)$ is a measurable space, and $A, B \in \mathcal{A}$. Prove that

$$\mu(A \cap B) + \mu(A \cup B) = \mu(A) + \mu(B).$$

5. If $(X, \mathcal{A}, \mu)$ is a measure space, define

$$\mu^*(A) = \inf\{\mu(B) : A \subseteq B, B \in \mathcal{A}\}$$

   for all subsets A of X. Show that μ^* is an outer measure. Show that each set in $\mathcal{A}$ is μ^*-measurable and μ^* agrees with the measure μ on $\mathcal{A}$.