Final Exam

1. A collection of subsets has the *finite intersection property* if every finite subcollection has nonempty intersection. Prove that a metric space X is compact if and only if every collection of closed sets with the finite intersection property has nonempty intersection.

2. A finite Borel measure μ on $\mathbb{R}^n$ is called *discrete* if there is a countable set $\{x_j\}$ of points in $\mathbb{R}^n$ and a collection of non-negative real numbers $\{c_j\}$ such that $\mu = \sum_j c_j \delta_{x_j}$. On the other hand, μ is called *continuous* if $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}^n$.

   (a) Show that any finite Borel measure μ on $\mathbb{R}^n$ can be uniquely decomposed as $\mu = \mu_d + \mu_c$, where μ_d is a discrete measure, μ_c is a continuous measure, and $\mu_d \perp \mu_c$.

   (b) Let μ be a finite Borel measure on $\mathbb{R}$. Show that μ is a continuous measure if and only if the function $x \mapsto \mu((-\infty, x])$ is continuous.

   (c) Let m be Lebesgue measure on $\mathbb{R}^n$. Using part (a), show that any finite Borel measure μ on $\mathbb{R}^n$ can be decomposed as $\mu = \mu_{ac} + \mu_d + \mu_{sc}$, where μ_{ac} is an absolutely continuous measure with respect to m, μ_d is discrete, and μ_{sc} is continuous but $\mu_{sc} \perp m$.[1]

3. Let $f : [a, b] \to \mathbb{R}$ be integrable, $a \in \mathbb{R}$, and

$$F(x) = \int_a^x f(y)\, dy, \quad a \leq x \leq b.$$

   Prove directly that F is of bounded variation and absolutely continuous on $[a, b]$.

4. Let $f : \mathbb{R}^n \to \mathbb{C}$ and $g : \mathbb{R}^n \to \mathbb{C}$ be Lebesgue measurable functions such that f is integrable and g is bounded. Show that the *convolution* $f * g$ defined by the formula

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y) g(x - y)\, dm(y)$$

   is well-defined (in the sense that the integrand on the right-hand side is integrable) and that $f * g$ is bounded and continuous.[2]

5. A function $f : [a, b] \to \mathbb{R}$ is said to be *Lipschitz continuous* if there exists $C > 0$ such that $|f(x) - f(y)| \leq C|x - y|$ for all $x, y \in [a, b]$. Show that every Lipschitz continuous function on $[a, b]$ is differentiable almost everywhere. (The same result is true in higher dimensions, and is known as the Radamacher differentiation theorem.) Give an example to show that Lipschitz continuous functions need not be differentiable at every point.

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[1] This decomposition is a useful refinement of the Lebesgue decomposition. Here, μ_{ac} is called the absolutely continuous part, μ_d is the pure point part, and μ_{sc} is the singular continuous part. This decomposition is unique, and, more generally, can be extended to regular complex Borel measures on $\mathbb{R}^n$.

[2] According to Terry Tao, "convolutions tend to be smoothing in nature; the convolution $f * g$ of two functions is usually at least as regular as, and often more regular than, either of the two factors f, g." Problem 4 is an example of this phenomenon.