Practice Midterm

The following is a list of problems I consider midterm-worthy. This list of problems should serve as a good place to start studying, and it should not be considered a comprehensive list of problems from the sections we've covered. YOU are responsible for studying all the sections to be covered on the midterm.

1. A random sample of size $n = 100$ is taken from an infinite population with mean $\mu = 40$ and variance $\sigma^2 = 9$. Use Chebyshev's inequality to estimate the probability that the value we obtain for $\bar{X}$ is between 37 and 43?
2. If S_1^2 and S_2^2 are the variances of independent random samples of sizes $n_1 = 10$ and $n_2 = 15$ from a normal population with $\sigma_1^2 = 12$ and $\sigma_2^2 = 18$, find $\mathbb{P}(S_1^2/S_2^2 > 1.16)$.
3. How many different samples of size $n = 10$ can be drawn from a finite population of size $N = 36$? What is the probability of each of these samples?
4. If X has F distribution with ν_1 and ν_2 degrees of freedom, show that $Y = \frac{1}{X}$ has F distribution with ν_2 and ν_1 degrees of freedom.
5. Let $X_1, \ldots, X_n$ be a random sample of size n from a uniform population on $[0, \beta]$ (i.e. $\alpha = 0$).
   (a) Show that $X_{(n)}$ is a consistent estimator of β.
   (b) Show that $X_{(1)}$ is a biased estimator of β and compute the bias.
6. Let $X_1, \ldots, X_n$ constitute a random sample of size n from a uniform population on $[0, 1]$. Compute the joint probability density function (jpdf) of $X_{(1)}$ and $X_{(n)}$.
7. Let X be a binomial random variable with parameters (n, θ). Show that $\frac{2X+3}{2n+1}$ is a biased estimator of the parameter θ. Is this estimator asymptotically unbiased?
8. Given a random sample of size n from a gamma population, use the method of moments to obtain formulas for estimating the parameters α and β
9. Given a random sample of size n from a uniform population with $\alpha = 3$, use the method of moments to obtain a formula for estimating the parameter β.
10. If $X_1, \ldots, X_n$ constitute a random sample of size n from an exponential population, show that $\bar{X}$ is a sufficient estimator of the parameter θ.

Select solutions and hints:

1. Recall that $\bar{X}$ has mean μ and variance σ^2/n. Thus, by Chebyshev's inequality,
$$\mathbb{P}(37 < \bar{X} < 43) = \mathbb{P}(|\bar{X} - \mu| < 3) = \mathbb{P}\left(|\bar{X} - \mu| < \frac{3}{\sigma/\sqrt{n}}\sigma/\sqrt{n}\right) \geq 1 - \frac{1}{n} = 0.99.$$

6. **Hint**: Compute the probability $\mathbb{P}(X_{(1)} > s, X_{(n)} \leq t)$.

7. Using that $\mathbb{E}[X] = n\theta$, we find
$$\mathbb{E}\left[\frac{2X+3}{2n+1}\right] = \frac{1}{2n+1}(2\mathbb{E}[X] + 3) = \frac{2n\theta + 3}{2n+1} \neq \theta.$$
So $\frac{2X+3}{2n+1}$ is a biased estimator of the parameter θ. However,
$$\lim_{n\to\infty} \mathbb{E}\left[\frac{2X+3}{2n+1}\right] = \theta,$$
and hence $\frac{2X+3}{2n+1}$ is an asymptotically unbiased estimator of θ.

8. To solve for both parameters, we will use both the first and second sample moments (m_1' and m_2', respectively). Based on the moments of the gamma distribution, we find
$$m_1' = \alpha\beta$$
and
$$m_2' = \alpha(\alpha+1)\beta^2.$$
Solving for α and β, we obtain the following two estimators:
$$\hat{\alpha} = \frac{(m_1')^2}{m_2' - (m_1')^2}$$
and
$$\hat{\beta} = \frac{m_2' - (m_1')^2}{m_1'}.$$

10. The joint probability density function (jpdf) of the random sample $X_1, \ldots, X_n$ is given by
$$f(x_1, \ldots, x_n; \theta) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-x_i/\theta} \chi(x_i),$$
where
$$\chi(x_i) = \begin{cases} 1, & \text{if } x_i > 0, \\ 0, & \text{otherwise.} \end{cases}$$
Thus, we decompose
$$f(x_1, \ldots, x_n; \theta) = \frac{1}{\theta^n} e^{-n\bar{x}/\theta} \prod_{i=1}^{n} \chi(x_i) = g(\theta, \bar{x}) h(x_1, \ldots, x_n),$$
where
$$g(\theta, \bar{x}) = \frac{1}{\theta^n} e^{-n\bar{x}/\theta}$$
and
$$h(x_1, \ldots, x_n) = \prod_{i=1}^{n} \chi(x_i).$$
By the factorization theorem, we conclude that $\bar{X}$ is a sufficient estimator of θ.