Practice Midterm

The following is a list of problems I consider midterm-worthy. This list of problems should serve as a good place to start studying, and it should not be considered a comprehensive list of problems from the sections we've covered. YOU are responsible for studying all the sections to be covered on the midterm.

1. Two cards are randomly selected from an ordinary playing deck. What is the probability that they form a blackjack? That is, what is the probability that one of the cards is an ace and the other one is either a ten, a jack, a queen, or a king?
2. An infinite sequence of independent trials is to be performed. Each trial results in a success with probability p and a failure with probability $1-p$. What is the probability that
   (a) exactly k successes occur in the first n trials?
   (b) at least k successes occur in the first n trials?
   (c) all trials result in successes?
3. Prove that if E and F are independent events, then
$$\mathbb{P}(E \cup F) = 1 - (1 - \mathbb{P}(E))(1 - \mathbb{P}(F)).$$
4. We toss n coins, and each one shows heads with probability p, independent of the others. Each coin which shows heads is tossed again. What is the probability distribution function of the number of heads resulting from the second round of tosses?
5. The joint probability density function (jpdf) of random variables X and Y is given by
$$f(x,y) = \begin{cases} \frac{C}{x^3y^4} & \text{if } x \geq 1, y \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$
   Compute the constant C and find the marginal densities of X and Y. Are X and Y independent?
6. Compute the moment generating function of a Bernoulli random variable with parameter θ.
7. Suppose the random variables X and Y take values $\{1,2,3\}$. If the joint probability distribution function of X and Y is
$$f(i,j) = \frac{i+j}{36}, \quad 1 \leq i,j \leq 3,$$
   are X and Y independent? Compute the covariance between X and Y.
8. Let X be a standard normal random variable. If $Y = e^X$, compute the density of Y.
9. Let X be a normal random variable with mean 1 and variance 12. Compute the following probabilities.
   (a) $\mathbb{P}(|X| > 7)$.
   (b) $\mathbb{P}(|X| \leq 12)$.
   (c) $\mathbb{P}(X < -1)$.
10. Recall that Γ is the gamma function defined by
$$\Gamma(\alpha) = \int_0^\infty e^{-x}x^{\alpha-1}dx.$$
   Show that $\Gamma(1/2) = \sqrt{\pi}$. Hint: Make use of the change of variables $y = \sqrt{2x}$. Then relate the resulting expression to the normal distribution.
11. A person tosses a fair coin until a tail appears for the first time. If the tail appears on the n-th flip, the person wins 2^n dollars. Let X denote the player's winnings. Compute the probability distribution function of X and show that $\mathbb{E}[X] = +\infty$.

12. In low-scoring team sports (e.g., soccer) the number of goals per game can often be approximated by a Poisson random variable. In the 2014-2015 season 8.16% of the games in the English Premier League ended in a scoreless tie (this means that there were no goals in the game). Estimate the percentage of games where exactly one goal was scored.
13. Let X and Y be iid random variables with mean μ and variance σ^2. Compute the mean and variance of $3X + 2Y - 5$. What is $\text{Cov}(-X, 3Y)$?

Select solutions and hints:

3. If E and F are independent, then it follows from one of your homework problems that E^c and F^c are independent. Therefore, using that E^c and F^c are independent, we compute

$$\mathbb{P}(E \cup F) = 1 - \mathbb{P}(E^c \cap F^c) = 1 - \mathbb{P}(E^c)\mathbb{P}(F^c).$$

   Since $\mathbb{P}(E^c) = 1 - \mathbb{P}(E)$ and $\mathbb{P}(F^c) = 1 - \mathbb{P}(F)$, the conclusion follows.

6. Let X be a Bernoulli random variable with parameter θ. Then the moment generating function $M_X(t)$ of X is

$$M_X(t) = \mathbb{E}[e^{Xt}] = \theta e^t + (1-\theta)e^0 = \theta e^t + 1 - \theta.$$

8. Let f_Y and F_Y be the pdf and cdf for the random variable Y. Then, for $a > 0$, we have

$$F_Y(a) = \mathbb{P}(Y \leq a) = \mathbb{P}(e^X \leq a) = \mathbb{P}(X \leq \ln a) = \Phi(\ln a),$$

   and hence

$$f_Y(a) = \frac{d}{da}F_Y(a) = \Phi'(\ln a)\frac{1}{a} = \frac{1}{\sqrt{2\pi}a}e^{-1/2(\ln a)^2}.$$

   Thus, we conclude that

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}y}e^{-\frac{1}{2}(\ln y)^2} & \text{if } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

10. Using the hint, we make the substitution $y = \sqrt{2x}$ and obtain

$$\Gamma(1/2) = \sqrt{2}\int_0^\infty e^{-y^2/2}dy = \frac{1}{\sqrt{2}}\int_{-\infty}^\infty e^{-y^2/2}dy = \frac{\sqrt{2\pi}}{\sqrt{2}} = \sqrt{\pi}.$$

   In the steps above, we used the density of the standard normal random variable to deduce that

$$\int_{-\infty}^\infty e^{-y^2/2}dy = \sqrt{2\pi}.$$