Practice Midterm

The following is a list of problems I consider midterm-worthy. This list of problems should serve as a good place to start studying, and it should not be considered a comprehensive list of problems from the sections we've covered. YOU are responsible for studying all the sections to be covered on the midterm.

1. A person tosses a fair coin until a tail appears for the first time. If the tail appears on the n-th flip, the person wins 2^n dollars. Let X denote the player's winnings. Compute the probability mass function of X and show that $\mathbb{E}[X] = +\infty$.

2. An urn contains 6 white and 8 black balls. We randomly choose 5 balls. If 2 of them are white and 3 are black, we stop. If not, we replace the balls in the urn and again randomly select 5 balls. This continues until exactly 2 of the 5 chosen are white. What is the probability that we shall make exactly n selections? What is the expected number of trials required before the selection contains exactly 2 white balls?

3. Let X be a normal random variable with mean 5 and variance 4. Compute the following probabilities. State your answer both in terms of the function Φ as well as a numerical value from the standard normal table.

   (a) $\mathbb{P}(|X| \leq 7)$.

   (b) $\mathbb{P}(X > 5)$.

   (c) $\mathbb{P}(X < 1)$.

   (d) $\mathbb{P}(3 < X \leq 5)$.

4. Two types of coins are produced by a factory: a fair coin and a biased one that comes up heads 55 percent of the time. We have one of these coins, but do not know whether it is a fair coin or a biased one. In order to ascertain which type of coin we have, we shall perform the following statistical test. We shall toss the coin 1000 times. If the coin lands on heads 535 or more times, then we shall conclude that it is a biased coin, whereas if it land on heads fewer than 535 times, then we shall conclude that it is a fair coin. If the coin is actually fair, what is the probability that we shall reach a false conclusion? What would it be if the coin were biased? You should use an appropriate approximation to compute the answers.

5. Let X have probability density function f_X. Find the probability density function of the random variable Y defined by $Y = 3X$.

6. For some constant c and some integer $n \geq 1$, the random variable X has the probability density function
$$f(x) = \begin{cases} cx^n, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$
   Compute the cumulative distribution function of X. Find the expectation and variance of X.

7. In low-scoring team sports (e.g., soccer) the number of goals per game can often be approximated by a Poisson random variable. In the 2014-2015 season 8.16% of the games in the English Premier League ended in a scoreless tie (this means that there were no goals in the game). Estimate the percentage of games where exactly one goal was scored.

8. Let Z be a standard normal random variable, and let g be a differentiable function.

   (a) Show that
$$\mathbb{E}[g'(Z)] = \mathbb{E}[Zg(Z)].$$

   (b) Compute $\mathbb{E}[Z^4]$.

9. Suppose exactly half of all watches produced by a certain factory are defective. A store buys a box with 400 watches produced by this factory. Assume this is a random sample from the factory.

(a) Write an expression for the exact probability that at least 215 of the 400 watches are defective.

(b) Approximate the probability that at least 215 of the 400 watches are defective.

Select solutions and hints:

1. X can take the values 2^n for $n = 1, 2, \ldots$. Since the coin is fair,

$$\mathbb{P}(X = 2^n) = \left(\frac{1}{2}\right)^{n-1} \left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^n.$$

Thus, the expectation of X is

$$\mathbb{E}[X] = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n 2^n = +\infty.$$

5. Let F_X denote the cumulative distribution function of X. Then the cumulative distribution function of Y is

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(3X \leq y) = \mathbb{P}(X \leq y/3) = F_X(y/3).$$

Therefore, the density f_Y of Y is

$$f_Y(y) = \frac{d}{dy} F_Y(y) = F'_X(y/3)\frac{1}{3} = f_X(y/3)\frac{1}{3}.$$

6. $\mathbb{E}[X] = \frac{n+1}{n+2}$ and

$$\text{Var}(X) = \frac{n+1}{n+3} - \left(\frac{n+1}{n+2}\right)^2.$$

7. Hint: Find the parameter λ of the Poisson random variable by computing the probability that the random variable is zero and comparing this to the percentage of games which ended in a scoreless tie.