Practice Midterm

The following is a list of problems I consider midterm-worthy. This list of problems should serve as a good place to start studying, and it should not be considered a comprehensive list of problems from the sections we've covered. YOU are responsible for studying all the sections to be covered on the midterm.

1. If 30 people are present in a room, what is the probability that at least two of them celebrate their birthday on the same day of the year?
2. Two cards are randomly selected from an ordinary playing deck. What is the probability that they form a blackjack? That is, what is the probability that one of the cards is an ace and the other one is either a ten, a jack, a queen, or a king?
3. Poker dice is played by simultaneously rolling 5 dice. Compute
   (a) $\mathbb{P}$(no two alike)
   (b) $\mathbb{P}$(one pair)
   (c) $\mathbb{P}$(two pair)
   (d) $\mathbb{P}$(three alike)
   (e) $\mathbb{P}$(five alike)
4. Consider a sample of size 3 drawn in the following manner: We start with an urn containing 5 white and 7 red balls. At each stage, a ball is drawn and its color is noted. The ball is then returned to the urn, along with an additional ball of the same color. Find the probability that the sample will contain exactly
   (a) 0 white balls;
   (b) 1 white ball;
   (c) 2 white balls;
   (d) 3 white balls.
5. Use a combinatorial argument to verify the following identity:
$$\sum_{k=1}^{n} \binom{n}{k} k^2 = 2^{n-2} n(n+1).$$
6. An infinite sequence of independent trials is to be performed. Each trial results in a success with probability p and a failure with probability $1-p$. What is the probability that
   (a) exactly k successes occur in the first n trials?
   (b) at least k successes occur in the first n trials?
   (c) all trials result in successes?
7. Roll two dice. What is the probability that the sum of the dice is 7 given that the product is 12? What is the probability that the product is 12 given that the sum is 7?
8. Prove that if E and F are independent events, then
$$\mathbb{P}(E \cup F) = 1 - (1 - \mathbb{P}(E))(1 - \mathbb{P}(F)).$$
9. An investor has 20,000 dollars to invest among 5 possible investments. Each investment must be in units of 1,000 dollars.
   (a) If the total 20,000 dollars is to be invested, how many different investment strategies are possible?
   (b) How many different investment strategies are possible if not all the money needs to be invested?

10. Verify the following identity:

$$\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}$$

for any $1 \leq k \leq n$.

Select solutions and hints:

1. 0.706
3. (a) $\mathbb{P}(\text{no two alike}) = 0.0926$.
   (b) $\mathbb{P}(\text{one pair}) = 0.4630$.
   (c) $\mathbb{P}(\text{two pair}) = 0.2315$.
   (d) $\mathbb{P}(\text{three alike}) = 0.1543$.
   (e) $\mathbb{P}(\text{five alike}) = 0.0008$.
5. Hint: consider a set of n people and argue that both sides of the identity represent the number of different selections of a committee, its chairperson, and its secretary (possibly the same as the chairperson).
8. If E and F are independent, then it follows from a proposition presented in class that E^c and F^c are independent. Therefore, using that E^c and F^c are independent, we compute

$$\mathbb{P}(E \cup F) = 1 - \mathbb{P}(E^c \cap F^c) = 1 - \mathbb{P}(E^c)\mathbb{P}(F^c).$$

   Since $\mathbb{P}(E^c) = 1 - \mathbb{P}(E)$ and $\mathbb{P}(F^c) = 1 - \mathbb{P}(F)$, the conclusion follows.
10. The identity can verified algebraically (Hint: start with the right-hand side). For a combinatorial argument: argue that the right-hand side counts the number of ways of selecting 2 people from a group of n people by first dividing the group into two subgroups of sizes k and $n - k$.