Practice Midterm

The following is a list of problems I consider midterm-worthy. This list of problems should serve as a good place to start studying, and it should not be considered a comprehensive list of problems from the sections we've covered. YOU are responsible for studying all the sections to be covered on the midterm.

1. A person tosses a fair coin until a tail appears for the first time. If the tail appears on the n-th flip, the person wins 2^n dollars. Let X denote the players winnings. Show that $\mathbb{E}[X] = +\infty$. Would you pay \$100 to play this game? This problem is known as the St. Petersburg paradox.
2. An urn contains 4 white and 4 black balls. We randomly choose 4 balls. If 2 of them are white and 2 are black, we stop. If not, we replace the balls in the urn and again randomly select 4 balls. This continues until exactly 2 of the 4 chosen are white. What is the probability that we shall make exactly n selections?
3. Let X be a normal random variable with mean 5 and variance 4. Compute the following probabilities. State your answer in terms of the function Φ.
   (a) $\mathbb{P}(|X| \leq 7)$.
   (b) $\mathbb{P}(X > 5)$.
   (c) $\mathbb{P}(X < 1)$.
4. Two types of coins are produced by a factory: a fair coin and a biased one that comes up heads 55 percent of the time. We have one of these coins, but do not know whether it is a fair coin or a biased one. In order to ascertain which type of coin we have, we shall perform the following statistical test. We shall toss the coin 1000 times. If the coin lands on heads 525 or more times, then we shall conclude that it is a biased coin, whereas if it land on heads fewer than 525 times, then we shall conclude that it is a fair coin. If the coin is actually fair, what is the probability that we shall reach a false conclusion? What would it be if the coin were biased? You should use an appropriate approximation to compute the answers.
5. Let X have probability density function f_X. Find the probability density function of the random variable Y defined by $Y = aX + b$ for some constants a and b with $a \neq 0$.
6. For some constant c and some integer $n \geq 1$, the random variable X has the probability density function
   $$f(x) = \begin{cases} cx^n, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$
   Find $\mathbb{E}[X]$ and $\mathbb{P}(X > x)$ for any $0 < x < 1$.
7. Roll two dice. Let X be the sum of the two dice and Y be the product of the two dice. Compute the joint probability mass function of X and Y.
8. Recall that Γ is the gamma function defined by
   $$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} dx.$$
   Show that $\Gamma(1/2) = \sqrt{\pi}$. Hint: Make use of the change of variables $y = \sqrt{2x}$. Then relate the resulting expression to the normal distribution.
9. A bin of 5 transistors is known to contain 2 that are defective. The transistors are to be tested, one at a time until the defective ones are identified. Denote by N_1 the number of tests made until the first defective transistor is identified and by N_2 the number of additional tests until the second defective transistor is identified. Find the joint probability mass function of N_1 and N_2.