Practice Midterm

The following is a list of problems I consider midterm-worthy. This list of problems should serve as a good place to start studying, and it should not be considered a comprehensive list of problems from the sections we've covered. YOU are responsible for studying all the sections to be covered on the midterm.

1. If 8 identical blackboards are to be divided among 4 schools, how many divisions are possible? How many if each school must receive at least 1 blackboard?
2. If n people are present in a room, what is the probability that no two of them the celebrate their birthday on the same day of the year?
3. Poker dice is played by simultaneously rolling 5 dice. Show that
   (a) $\mathbb{P}(\text{no two alike}) = 0.0926$.
   (b) $\mathbb{P}(\text{one pair}) = 0.4630$.
   (c) $\mathbb{P}(\text{two pair}) = 0.2315$.
   (d) $\mathbb{P}(\text{three alike}) = 0.1543$.
   (e) $\mathbb{P}(\text{five alike}) = 0.0008$.
4. Use a combinatorial argument to verify the following identity:
$$\sum_{k=1}^{n} \binom{n}{k} k^2 = 2^{n-2} n(n+1).$$
   Hint: consider a set of n people and argue that both sides of the identity represent the number of different selections of a committee, its chairperson, and its secretary (possibly the same as the chairperson).
5. An infinite sequence of independent trials is to be performed. Each trial results in a success with probability p and a failure with probability $1-p$. What is the probability that
   (a) at least k successes occur in the first n trials?
   (b) all trials result in successes?
6. Roll two dice. What is the probability that the sum of the dice is 7 given that the product is 12?
7. Let X be the winnings of a gambler. Let $p(i)$ be the probability mass function of X and suppose that
$$p(0) = 1/3, \quad p(1) = p(-1) = 13/55, \quad p(2) = p(-2) = 1/11, \quad p(3) = p(-3) = 1/165.$$
   Compute the conditional probability that the gambler wins i, $i = 1, 2, 3$, given that he wins a positive amount.
8. Two fair dice are rolled. Let X equal the product of the two dice. Compute the probability mass function of X and find $\mathbb{E}[X]$.
9. Prove that if E and F are mutually exclusive, then
$$\mathbb{P}(E|E \cup F) = \frac{\mathbb{P}(E)}{\mathbb{P}(E) + \mathbb{P}(F)}.$$
10. An investor has 20,000 dollars to invest among 5 possible investments. Each investment must be in units of 1,000 dollars.
    (a) If the total 20,000 dollars is to be invested, how many different investment strategies are possible?
    (b) How many different investment strategies are possible if not all the money needs to be invested?