Homework #5

1. Give an example of each request or give an argument explaining that such a request is impossible.
   (a) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.
   (b) A sequence which has no convergent subsequence.
   (c) A sequence which contains a bounded subsequence but contains no subsequences which converge.
   (d) A sequence that contains subsequences converging to every point in the infinite set \( \{1, 1/2, 1/3, 1/4, \ldots\} \).
   (e) A Cauchy sequence that is not monotone.
   (f) A Cauchy sequence with no convergent subsequence.

2. Prove that the infinite series
   \[ \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \]
   converges to 1.

3. Let \((a_n)\) and \((b_n)\) be Cauchy sequences. Prove that \((c_n)\) and \((d_n)\) are also both Cauchy sequences, where
   \[ c_n = a_n + b_n \]
   and
   \[ d_n = |a_n - b_n| \]

4. Let \((a_n)\) be a bounded sequence with the property that every convergent subsequence converges to the same limit \(a \in \mathbb{R}\). Show that \((a_n)\) must converge to \(a\).

5.* Consider the sequence \(a_n = \left(1 + \frac{1}{n}\right)^n\).
   (a) Show that \((a_n)\) is an increasing sequence.
   (b) Prove that \(1 \leq a_n \leq 3\) for all \(n \in \mathbb{N}\).
   (c) Use the Monotone Convergence Theorem to conclude that \((a_n)\) converges. The limit of \((a_n)\) is called *Euler’s number* and is typically denoted by \(e\).