Homework #5

1. Give an example of each request or give an argument explaining that such a request is impossible.
   (a) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.
   (b) A sequence which has no convergent subsequence.
   (c) A sequence which contains a bounded subsequence but contains no subsequences which converge.
   (d) A sequence that contains subsequences converging to every point in the infinite set $\{1, 1/2, 1/3, 1/4, \ldots\}$.
   (e) A Cauchy sequence that is not monotone.
   (f) A Cauchy sequence with no convergent subsequence.
2. Prove that the infinite series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges to 1.
3. Let (a_n) and (b_n) be Cauchy sequences. Prove that (c_n) and (d_n) are also both Cauchy sequences, where $c_n = a_n + b_n$ and $d_n = |a_n - b_n|$.
4. Let (a_n) be a bounded sequence with the property that every convergent subsequence converges to the same limit $a \in \mathbb{R}$. Show that (a_n) must converge to a.

5.* Consider the sequence $a_n = \left(1 + \frac{1}{n}\right)^n$.
   (a) Show that (a_n) is an increasing sequence.
   (b) Prove that $1 \leq a_n \leq 3$ for all $n \in \mathbb{N}$.
   (c) Use the Monotone Convergence Theorem to conclude that (a_n) converges. The limit of (a_n) is called *Euler's number* and is typically denoted by e.