Select solutions to Homework #4

11.6 (a) By the triangle inequality,

$$|x| = |x - y + y| \leq |x - y| + |y|. \tag{1}$$

Similarly,

$$|y| = |y - x + x| \leq |y - x| + |x|.$$

Thus, rearranging slightly, we have

$$-|x - y| \leq |x| - |y| \leq |x - y|.$$

Here we used the property that $|x - y| = |y - x|$. By Theorem 11.9, we conclude that $||x| - |y|| \leq |x - y|$. This inequality is usually known as the **reverse triangle inequality**.

(b) By (1), $|x| \leq |x - y| + |y| < c + |y|$ whenever $|x - y| < c$.

(c) Assume $|x - y| < \varepsilon$ for all $\varepsilon > 0$. By Theorem 11.7, this implies that that $|x - y| = 0$. Hence, by Theorem 11.9, we obtain $x = y$.

12.10 (a) This proof was inspired by an idea of Zack Thoutt. Fix $x, y \in \mathbb{R}$ with $x < y$. Define

$$N = \{n \in \mathbb{N} : \text{there exists at least } n \text{ rational numbers in the interval } (x, y)\}.$$

By induction, we will show that $N = \mathbb{N}$. Theorem 12.12 implies that $1 \in N$. Assume $n \in N$. Then there exists rational numbers $r_1, \ldots, r_n$ such that $x < r_1 < \cdots < r_n < y$. By Theorem 12.12, there exists a rational number r_{n+1} such that $r_n < r_{n+1} < y$, and hence $n + 1 \in N$. By the principle of induction, we conclude that $N = \mathbb{N}$.
To complete the proof, assume there only exists a finite number n of rational numbers between x and y. By the proceeding argument, $n+1 \in N$, which contradicts the assumption that there were only n rational numbers between x and y. Therefore, we conclude that there must be an infinite number of rational numbers between x and y.

13.7 (a) Take $S = \{1/n : n \in \mathbb{N}\}$.

(c) Take $S = (1, 2) \cup (2, 3)$.

(e) Try $S = \mathbb{Q}$.

13.9 (b) Let x be a boundary point of S. Then $N(x, \varepsilon) \cap S \neq \emptyset$ and $N(x, \varepsilon) \cap (\mathbb{R} \setminus S) \neq \emptyset$ for all $\varepsilon > 0$. Suppose x is not an accumulation point of S (for if it is an accumulation point, we are done). Notice that if $x \notin S$, then $N^*(x, \varepsilon) \cap S \neq \emptyset$ for every $\varepsilon > 0$, and x would be an accumulation point of S. Thus, it must be the case that $x \in S$. But, by definition, this implies that x is an isolated point, completing the proof.