MTH 321 Reference

Kinds of Differential Equations

Autonomous: Does not depend on time.
Linear: If $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$ are solutions, so is $c\mathbf{u}_1(t) + d\mathbf{u}_2(t)$. Only first powers of the variables appear in the equations.
Homogeneous: Of a linear equation, the constant term $(q(t))$ is 0.
Associated Homogeneous Equation: An equation obtained from a linear differential equation by setting the constant term to 0.
n**th-order:** Contains nth derivatives.
n**-dimensional:** Contains n variables.

$$\frac{dy}{dt} = f(t, y) \qquad \text{First-order Equation}$$

$$\frac{dy}{dt} = f(y)g(t) \qquad \text{Separable Equation}$$

$$\frac{dy}{dt} = f(y) \qquad \text{First-order Autonomous Equation}$$

$$\frac{dy}{dt} = p(t)y + q(t) \qquad \text{First-order Linear Equation}$$

$$\frac{dy}{dt} = p(t)y \qquad \text{Homogeneous Linear Equation}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix} \qquad \text{Second-Dimensional System}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{Second-Dimensional Linear System}$$

$$y'' = f(t, y, y'') \qquad \text{Second-Order Equation}$$

$$y'' + py' + qy = f(t) \qquad \text{Second-Order Linear Equation}$$

$$y'' + py' + qy = 0 \qquad \text{Second-Order Homogeneous Linear Equation}$$

Solutions

Equilibria: Solutions to an equation such that the derivative is always zero, i.e. constant solutions.
Stable Equilibrium: An equilibrium that all "nearby" solutions converge to.
Unstable Equilibrium: An equilibrium that not all nearby solutions converge to.
Semistable Equilibrium: A subtype of unstable equilibria that some nearby solutions converge to and some nearby solutions diverge from.
Nullcline: A curve along which one or more derivatives of a system's variables are 0.
Particular Solution: A solution to a differential equation for some particular initial conditions.
General Solution: A solution to a differential equation for all initial conditions, written in terms of a few constants.
Homogeneous Solution: The general solution to the associated homogeneous equation of a linear differential equation.

Simple Solutions.

$$\frac{dy}{dt} = ky \implies y(t) = Ce^{kt}$$
$$\frac{dy}{dt} = f(y)g(t) \implies \int \frac{1}{f(y)}\,dy = \int g(t)\,dt$$
$$\frac{dy}{dt} = p(t)y \implies y(t) = Ce^{\int p(t)\,dt}$$

General solution to first-order linear equation by variation of parameters:

$$\frac{dy}{dt} = p(t)y + q(t) \implies y(t) = e^{\int p(t)\,dt} \int q(t)e^{-\int p(t)\,dt}\,dt + ke^{\int p(t)\,dt}$$

Constant-Coefficient Linear Systems. In general:

$$\mathbf{u}' = \mathbf{A}\mathbf{u}$$

where:

$$\mathbf{u} = \left(xy\right)$$

and

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

If the two solutions to the equation

$$\lambda^2 - (a+d)\lambda + ad - bc = 0$$

are:

(1) Both real, different, and positive, then unstable node
(2) Both real and different, one positive, one negative, then saddle node
(3) Both real, different, and negative, then stable node
(4) Both equal and positive, then outward degenerate spiral node
(5) Both equal and negative, then inward degenerate spiral node
(6) Imaginary, with positive real part, then outward spiral node
(7) Imaginary, with zero real part, then ellipses
(8) Imaginary, with negative real part, then inward spiral node

Laplace Transformations

$$\begin{aligned}
\mathcal{L}\left\{cf(t)+dg(t)\right\} &= cF(s)+dG(s) \\
\mathcal{L}\left\{1\right\} &= \frac{1}{s} \\
\mathcal{L}\left\{t^n\right\} &= \frac{n!}{s^{n+1}} \\
\mathcal{L}\left\{\cos(at)\right\} &= \frac{s}{s^2+a^2} \\
\mathcal{L}\left\{\sin(at)\right\} &= \frac{a}{s^2+a^2} \\
\mathcal{L}\left\{tf(t)\right\} &= -\frac{d}{ds}F(s) \\
\mathcal{L}\left\{t^n f(t)\right\} &= (-1)^n\frac{d^n}{ds^n}F(s) \\
\mathcal{L}\left\{e^{at}f(t)\right\} &= F(s-a) \\
\mathcal{L}\left\{h_a(t)f(t-a)\right\} &= e^{-sa}F(s) \\
\mathcal{L}\left\{\delta_a(t)\right\} &= e^{-sa} \\
\mathcal{L}\left\{y'(t)\right\} &= sY(s)-y(0) \\
\mathcal{L}\left\{y''(t)\right\} &= s^2Y(s)-sy(0)-y'(0)
\end{aligned}$$