ANALYSIS 6310 REFERENCE

"Well, this is easy." – Sergei Kuznetsov

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1. Definitions

There are more than a few. I'll use $\overline{R}$ for $\mathbb{R} \cup \{\pm\infty\}$.

Partition: A *partition* of $[a,b]$ is a collection of points $\{x_0, \ldots, x_m\}$ so that $x_0 = a, x_m = b$ and $x_{i-1} < x_i$ for $i = 1, \ldots, m$.

Variation: The *variation of f over* $[a,b]$ is defined by

$$V = V[f; a, b] = \sup_\Gamma S_\Gamma,$$

where S_Γ is the sum

$$\sum_{i=1}^{m} |f(x_i) - f(x_{i-1})|$$

and the supremum is taken over all partitions Γ of $[a,b]$. Intuitively, the variation is how much f moves up and down over the interval $[a,b]$.

Function of Bounded Variation: one with $V[f;a,b]$ finite.

Function of Unbounded Variation: one with $V[f;a,b] = +\infty$.

Positive Variation: The *positive variation of f over* $[a,b]$ is defined by

$$P = P[f; a, b] = \sup_\Gamma P_\Gamma$$

where

$$P_\Gamma = \sum_{i=1}^{m} (f(x_i) - f(x_{i-1}))^+$$

and the supremum is taken over all partitions Γ of $[a,b]$. Intuitively, the positive variation is how much f moves up over the interval $[a,b]$.

Negative Variation: The *negative variation of f over* $[a,b]$ is defined by

$$N = N[f; a, b] = \sup_\Gamma N_\Gamma$$

where

$$N_\Gamma = \sum_{i=1}^{m} (f(x_i) - f(x_{i-1}))^-$$

and the supremum is taken over all partitions Γ of $[a,b]$. Intuitively, the negative variation is how much f moves down over the interval $[a,b]$.

Rectifiable Curve: Intuitively, a curve with finite length. Formally, the length L of a curve C with coordinate functions $\phi : [a,b] \to \mathbb{R}$ and $\psi : [a,b] \to \mathbb{R}$ is defined as the supremum (over Γ) of the sums

$$L(\Gamma) = \sum_{i=1}^{m} \sqrt{(\phi(t_i) - \phi(t_{i-1}))^2 + (\psi(t_i) - \psi(t_{i-1}))^2}$$

where Γ is a partition $\{t_0 = a, \ldots, t_m = b\}$ of $[a, b]$. We say C is a *rectifiable curve* if L is finite.

Norm of a Partition: The norm $|\Gamma|$ of a partition Γ is the length of the largest interval of Γ. That is, if $\Gamma = \{x_0, \ldots, x_n\}$, $|\Gamma| = \max_i\{x_i - x_{i-1}\}$.

Riemann-Stieltjes Integral: Intuitively, the Riemann integral with a change of variables built in. Formally, let f and ϕ be two functions which are defined and finite on a finite interval $[a, b]$. If $\Gamma = \{a = x_0 < x_1 < \cdots < x_m = b\}$ is a partition of $[a, b]$, we arbitrarily select intermediate points $\{\xi_i\}_{i=1}^m$ satisfying $x_{i-1} \le \xi_i \le x_i$, and write

$$R_\Gamma = \sum_{i=1}^{m} f(\xi_i)(\phi(x_i) - \phi(x_{i-1})).$$

R_Γ is called a *Riemann-Stieltjes sum* for Γ, and of course deponds on ξ_i, f, ϕ, etc, but we don't bother to indicate this dependence in our notation. Then, if $I = \lim_{|\Gamma|\to 0} R_\Gamma$ exists and is finite, that is, if given $\epsilon > 0$ there is a $\delta > 0$ such that $|I - R_\Gamma| < \epsilon$ for any Γ satisfying $|\Gamma| < \delta$, then I is called *the Riemann-Stieltjes integral of f with respect to ϕ on* $[a, b]$, and denoted

$$I = \int_a^b f(x)\, d\phi(x) = \int_a^b f\, d\phi.$$

Step Function: A function ϕ whose domain may be partitioned into finitely many intervals so that ϕ is constant on each interval.

Lebesgue Outer Measure: Intuitively, the smallest volume of intervals that cover E. Formally, let E be a subset of $\mathbb{R}^n$. Cover E by a *countable* collection S of n-dimensional closed intervals I_k, and let

$$\sigma(S) = \sum_{I_k \in S} v(I_k),$$

where v is the n-volume of the interval I_k. The *Lebesgue outer measure* (or *exterior measure*) of E, denoted $|E|_e$, is defined by

$$|E|_e = \inf \sigma(S),$$

where the infimum is taken over all such covers S of E.

Lebesgue Measurable Set: Intuitively, a set which is well approximated by a collection of intervals. Formally, a subset E of $\mathbb{R}^n$ is said to be *Lebesgue measurable*, or simply *measurable*, if given $\epsilon > 0$, there exists an open set G such that

$$E \subset G \text{ and } |G - E|_\epsilon < \epsilon.$$

(Note that open sets are precisely those that can be expressed as a countable union of open intervals, hence the intuitive interpretation.)

Lebesgue Measure: If E is Lebesgue measurable, we define its *measure* $|E|$ to be its outer measure $|E|_e$. Intuitively, this is the volume of E.

σ-Algebra: A collection of sets Σ that is closed under complements, countable unions, and countable intersections. (The first two properties imply the third.)

Borel Set: A set obtainable by complements, countable unions, and countable intersections from open sets in finitely many steps. Alternatively, a member of the σ-algebra generated by the open subsets of $\mathbb{R}^n$.

Almost Everywhere: A property is said to hold *almost everywhere* (or a.e if we're feeling lazy) on a set E, if the set of points of E where it does *not* hold has measure zero.

Measurable Function: A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ so that the preimage of each interval $(a, \infty]$ is a measurable set for each finite $a \in \mathbb{R}$. Intuitively, these are the functions so that the Lebesgue integral makes sense.

Upper-semicontinuous Function: Intuitively, a function whose limsups are not too large. Formally, $f : E \subset \mathbb{R}^n \to \overline{\mathbb{R}}$ is upper semicontinuous (or usc if we're feeling lazy) at $\mathbf{x}_0$ if

$$\limsup_{\mathbf{x}\to\mathbf{x}_0; x\in E} f(\mathbf{x}) \leq f(\mathbf{x}_0).$$

Alternatively, we have exactly one half of the ϵ-δ definition of continuity: f is usc at $\mathbf{x}_0$ if for all $\epsilon > 0$ there exists $\delta > 0$ so that for all $\mathbf{x} \in E$ with $|\mathbf{x} - \mathbf{x}_0| < \delta$ it follows that

$$f(\mathbf{x}) - f(\mathbf{x}_0) < \epsilon.$$

Lower-semicontinuous Function: Intuitively, a function whose liminfs are not too small. Formally, $f : E \subset \mathbb{R}^n \to \overline{\mathbb{R}}$ is *lower semicontinuous* (or lsc if we're feeling lazy) at $\mathbf{x}_0$ if

$$\liminf_{\mathbf{x}\to\mathbf{x}_0; x\in E} f(\mathbf{x}) \geq f(\mathbf{x}_0).$$

Alternatively, we have the other half of the ϵ-δ definition of continuity: f is lsc at $\mathbf{x}_0$ if for all $\epsilon > 0$ there exists $\delta > 0$ so that for all $\mathbf{x} \in E$ with $|\mathbf{x} - \mathbf{x}_0| < \delta$ it follows that

$$-\epsilon < f(\mathbf{x}) - f(\mathbf{x}_0).$$

Property $\mathscr{C}$: Intuitively, discontinuous in only a set of arbitrarily small measure. Formally, f has property $\mathscr{C}$ on E if given $\epsilon > 0$, there is a closed set $F \subset E$ such that

(1) $|E - F| < \epsilon$
(2) f is continuous relative to F.

If E is measurable, then this is equivalent to f being measurable on E. (This definition appears to be endemic to our textbook.)

Convergence in Measure: A sequence of functions f_k is said to converge in measure to f on E (written $f_k \xrightarrow{m} f$) if for every $\epsilon > 0$,

$$\lim_{k\to\infty} |\{x \in E : |f(\mathbf{x}) - f_k(\mathbf{x})| > \epsilon\}| = 0.$$

Intuitively, the size of the set where f_k is far from f can be made to have arbitrarily small measure by choosing k large.

Lebesgue Integral: This is defined in two steps. For the first step, let $f : E \subset \mathbb{R}^n \to \overline{R}$ be a non-negative function. Define

$$R(f, E) = \{(\mathbf{x}, y) \in \mathbb{R}^{n+1} : \mathbf{x} \in E, y \in \mathbb{R}, 0 \leq y \leq f(\mathbf{x})\}.$$

This is the region between 0 and the graph of f. If $R(f, E)$ is measurable, we define the *Lebesgue integral of f orever E* as

$$|R(f, E)|_{(n+1)} = \int_E f(\mathbf{x})\, d\mathbf{x}.$$

In the case that f is not non-negative, we define

$$\int_E f(\mathbf{x})\,d\mathbf{x} = \int_E f^+\,d\mathbf{x} - \int_E f^-\,d\mathbf{x}$$

provided that at least one of the integrals on the right is finite.

Simple Function: A function f is *simple* if it takes finitely many values.

Integrable Function: A function $f : E \subset \mathbb{R}^n \to \overline{\mathbb{R}}$ is *integrable* if $\int_E f$ exists and is finite.

L^p Space: $L^p(E)$, $0 < p < \infty$ is the set of functions $f : E \to \overline{\mathbb{R}}$ so that $|f|^p$ is integrable over E. In particular, if f is integrable, $f \in L(E)$.

Equimeasurable: Two functions $f, g : E \subset \mathbb{R}^n \to \overline{\mathbb{R}}$ are *equimeasurable* or *equidistributed* if

$$|\{\mathbf{x} \in E : f(\mathbf{x}) > \alpha\}| = |\{\mathbf{x} \in E : f(\mathbf{x}) > \alpha\}|$$

for all α. In the notation of §5.4, $\omega_{f,E} = \omega_{g,E}$.

Convolution: If f and g are measurable functions in $\mathbb{R}^n$, their convolution $(f * g)(\mathbf{x})$ is defined by

$$(f * g)(\mathbf{x}) = \int_{\mathbb{R}^n} f(\mathbf{x} - \mathbf{t})g(\mathbf{t})\,d\mathbf{t}.$$

Set Function: A *set function* is a real-valued function F defined on a σ-algebra Σ of measurable sets such that

(1) $F(E)$ is finite for every $E \in \Sigma$,
(2) F is *countably additive*; i.e., if $E = \cup_k E_k$ is a union of disjoint $E_k \in \Sigma$, then

$$F(E) = \sum_k F(E_k).$$

Indefinite Integral: If $f \in L(A)$, where A is a measurable subset of $\mathbb{R}^n$, the *indefinite integral of f* is defined to be the set function

$$F(E) = \int_E f,$$

where E is any measurable subset of A.

Continuous Set Function: A set function $F(E)$ is called *continuous* if $F(E) \to 0$ as the diameter $\sup\{|\mathbf{x} - \mathbf{y}| : \mathbf{x}, \mathbf{y} \in E\}$ tends to 0; that is, $F(E)$ is continuous if, given $\epsilon > 0$, there exists $\delta > 0$ such that $|F(E)| < \epsilon$ whenever the diameter of E is less than δ.

Absolutely Continuous Set Function: A set function F is called *absolutely continuous* if $F(E)$ tends to zero as the measure of E tends to zero. If you like ϵs and δs, F is absolutely continuous if given $\epsilon > 0$, there exists $\delta > 0$ such that $|F(E)| < \epsilon$ whenever the measure of E is less than δ.

Hardy-Littlewood Maximal Function: If f is a function defined on $\mathbb{R}^n$ and integrable over every cube Q, we define the *Hardy-Littlewood maximal function of f* by

$$f^*(\mathbf{x}) = \sup \frac{1}{|Q|} \int_Q |f(\mathbf{y})| d\mathbf{y}$$

where the supremum is taken over all Q with edges parallel to the coordinate axes and center $\mathbf{x}$. Other sources tend to define this in terms of balls centered at $\mathbf{x}$ rather than cubes.

Weak $L(\mathbb{R}^n)$: A function f belongs to *weak* $L(\mathbb{R}^n)$ if there is a constant c independent of α so that

$$|\{\mathbf{x} \in \mathbb{R}^n : |f(\mathbf{x})| > \alpha\} \leq \frac{c}{\alpha}$$

for all $\alpha > 0$. These are functions that obey Tchebyshev's Inequality except they get the constant wrong.

Locally Integrable: A function f is *locally integrable* on E if it is integrable over every bounded measurable subset of E.

Point of Density: $\mathbf{x}$ is a *point of density of* E if

$$\lim_{Q \searrow \mathbf{x}} \frac{|E \cap Q|}{|Q|} = 1.$$

Point of Dispersion: $\mathbf{x}$ is a *point of dispersion of* E if

$$\lim_{Q \searrow \mathbf{x}} \frac{|E \cap Q|}{|Q|} = 0.$$

Cover in the Sense of Vitali: A family K of cubes is said to cover a set E in the *Vitali sense* if for every $\mathbf{x} \in E$ and $\eta > 0$, there is a cube in K containing $\mathbf{x}$ whose diameter is less than η.

Absolutely Continuous Function: A finite function f on a finite interval $[a, b]$ is said to be *absolutely continuous* if given $\epsilon > 0$, there exists $\delta > 0$ such that for any collection $\{[a_i, b_i]\}$ (finite or countable) of nonoverlapping subintervals of $[a, b]$,

Singular Function: A function f is *singular* on $[a, b]$ if f' is zero a.e. in $[a, b]$.

Convex Function: Let ϕ be defined and finite on an interval (a, b). We say ϕ is *convex* in (a, b) if for every $[x_1, x_2]$ in (a, b), the graph of ϕ on $[x_1, x_2]$ lies on or below the line segment connecting the points $(x_1, \phi(x_2))$ and $(x_2, \phi(x_2))$. In other words, the region above the graph of ϕ is convex.

2. Frequently Cited and Otherwise Important Theorems

3. Examples

3.1. **The Dirichlet Function.** This is the function defined on $[0, 1]$ by

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

This function is integrable, but not Riemann-integrable. It is defined on a finite interval, but has unbounded variation.

3.2. **The Cantor Set.** You know what this is. It is uncountable, but has measure 0.

3.3. **The Cantor-Lebesgue Function.** Singular, but not constant. Has bounded variation.