INTRO TO SHEAVES AND ABELIAN CATEGORIES

SEBASTIAN BOZLEE

1. Intro

Today I'd like to talk about what sheaves are, and why we can do homological algebra on them.

2. Sheaves

Fix a topological space X and consider the set

$$C(X) = \{f : X \to \mathbb{R} \mid f \text{ is continuous}\}.$$

These functions have one key property, namely an element f of $C(X)$ can be specified by defining it on an open cover of X. (Picture: two functions defined on halves of an interval. Since they agree on overlaps, they define a single continuous function from the interval to $\mathbb{R}$.)

Our goal will be to define a mathematical object which abstracts this idea of "data on a topological space that can be defined on an open cover." We shall call such objects sheaves.

In order to make sense of defining f on some open subset $U \subseteq X$, we need the notion of function restriction, which is a group homomorphism $-|_U : C(X) \to C(U)$. So as a first step toward defining a sheaf we should abstract "data on a topological space that can be restricted to smaller open subsets." The definition is surprisingly simple.

Definition. Let $\text{Open}(X)$ be the category whose elements are the open subsets of X, with an arrow $V \to U$ if and only if $V \subseteq U$.

A **Presheaf** (of abelian groups) on X is a contravariant functor $\mathscr{F} : \text{Open}(X)^{op} \to \mathbf{Ab}$.

What does this mean? It is worth expanding the definition.

Definition. (Expanded) A **Presheaf** on a topological space X consists of the data

(i) For each open set U of X, an abelian group $\mathscr{F}(U)$.
(ii) For each inclusion of open sets $V \subseteq U$, a **restriction morphism** $\rho_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$.

subject to the conditions

(i) $\rho_{UU} = \text{id}_U$ for all U.
(ii) For all $W \subseteq V \subseteq U$, $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$.

Example: Let X be a topological space. We may define a presheaf C on X by taking, for each open subset U of X,

$$C(U) = \{f : U \to \mathbb{R} \mid f \text{ is continuous.}\}$$

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We take the group homomorphisms $\rho_{UV} : C(U) \to C(V)$ to be defined by $\rho_{UV}(f) = f|_V$. This defines a sheaf C on X.

If X is a smooth manifold, we can replace "continuous" by "differentiable" or "C^∞" and obtain a presheaf again.

This example plays such a strong role in our mental image of sheaves that when $f \in \mathscr{F}(U)$, for **any** presheaf $\mathscr{F}$, we will often use the notation $f|_V$ instead of $\rho_{UV}(f)$. Such an element f is called a **section** of $\mathscr{F}$.

Example: If G is any abelian group, we can also define the **constant presheaf** $\mathscr{F}_G$ by

$$\mathscr{F}_G(U) = \{f : U \to G \mid f \text{ is constant}\}$$

with restriction maps the restriction of functions.

Definition. A **morphism of presheaves** $\varphi : \mathscr{F} \to \mathscr{G}$ is a natural transformation from $\mathscr{F} \to \mathscr{G}$.

Again, this is worth expanding:

Definition. (Expanded) A **morphism of presheaves** $\varphi : \mathscr{F} \to \mathscr{G}$ is a collection of group homomorphisms $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$, one for each open subset U of X, so that for all $V \subseteq U$, the diagram

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "\varphi_U"'] \arrow[d, "\rho^{\mathscr{F}}_{UV}"'] & \mathscr{G}(U) \arrow[d, "\rho^{\mathscr{G}}_{UV}"] \\ \mathscr{F}(V) \arrow[r, "\varphi_V"'] & \mathscr{G}(V) \end{tikzcd}$$

commutes.

Altogether, we get a category of presheaves of abelian groups on X, denoted PreSh(X).

Now we are ready to define a sheaf.

Definition. A **sheaf** (of abelian groups) on a topological space X is a presheaf $\mathscr{F}$ satisfying the **sheaf axiom**:

Let $\{U_\alpha\}_{\alpha \in A}$ be any collection of open sets of X and let U be their union. Then, given sections $f_\alpha \in \mathscr{F}(U_\alpha)$ for each α so that

$$f_\alpha|_{U_\alpha \cap U_\beta} = f_\beta|_{U_\alpha \cap U_\beta}$$

for each $\alpha, \beta \in A$, there exists a unique element $f \in \mathscr{F}(U)$ so that

$$f|_{U_\alpha} = f_\alpha$$

for all $\alpha \in A$.

Example: The presheaf of continuous functions on X is a sheaf.

(Picture of functions on two open subsets of an interval. Point out that we glue only when we agree on overlaps. The reason gluing works is that a function is continuous if and only if it is continuous in a neighborhood of each point.)

Example: The constant presheaf $\mathscr{F}_{\mathbb{R}}$ is **not** a sheaf.

(Picture of constant functions on the disjoint union of two intervals. The reason that gluing fails is that being a constant function is not the same as being constant in a neighborhood of each point.)

An analogous definition that works is the **constant sheaf** $\underline{G}$ defined by

$$\underline{G}(U) = \{f : U \to G \mid f \text{ is continuous}\}$$

where G is given the discrete topology. (Equivalently, $\underline{G}(U)$ is the set of locally constant functions from U to G.)

Definition. A morphism between sheaves $\mathscr{F}$ and $\mathscr{G}$ is a morphism of presheaves between $\mathscr{F}$ and $\mathscr{G}$.

The resulting category of sheaves of abelian groups on X is denoted $\mathrm{Sh}(X)$

Remark: Although we have used the ability to define something locally as our motivation for sheaves, we gain more by thinking in terms of sheaves than merely having this property of functions present in our minds. Thinking in terms of sheaves also encourages us to think of nice functions on all open subsets of the space X, rather than merely the globally defined ones. This often gives us access to far more information.

Remark: We can often think of sheaves of abelian groups as collections of things that are metaphorically nice functions on our space. This is not an empty idea. In algebraic geometry, we think of the elements of arbitrary rings as functions on certain spaces, even though we probably shouldn't think of these ring elements as literally functions. One reason this is excusable is that they are enough like functions to form a sheaf.

3. Additive Categories

Now that we know what sheaves are, let us suppose that we want to do homological algebra on these things. This means that we need notions of exactness, kernels, quotients, images, and homology. As a first step in this direction, we will define an additive category.

Definition. An **additive category** is a category $\mathcal{A}$ so that

(i) There is a zero object. (An object which is both initial and terminal.)
(ii) All pairwise products exist.
(iii) Each set $\mathrm{Hom}_{\mathcal{A}}(B, C)$ is imbued with the structure of an abelian group so that given $f : A \to B$, $g, g' : B \to C$ and $h : C \to D$,

$$h \circ (g + g') = h \circ g + h \circ g'$$

and

$$(g + g') \circ f = g \circ f + g' \circ f.$$

Example: R-mod, Ab, $\mathrm{Ch}(\mathcal{A})$, Presheaves, Sheaves are additive categories.

By $\mathrm{Ch}(\mathcal{A})$, I mean the category of chain complexes whose elements come from an additive category $\mathcal{A}$. ($d \circ d = 0$ makes sense because each Hom-set is an abelian group.) The morphisms are chain maps.

Definition. A functor $F : \mathcal{A} \to \mathcal{B}$ between additive categories is an **additive functor** if for each $X, Y \in \mathcal{A}$, the map

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(X,Y) &\to \mathrm{Hom}_{\mathcal{B}}(F(X),F(Y)) \\ f &\mapsto F(f) \end{aligned}$$

is a group homomorphism.

Definition. A **kernel** of a map $f : A \to B$ is a morphism $\ker(f) : K \to A$ so that $f \circ \ker(f) = 0$ and given any map $g : Z \to A$ so that $f \circ g = 0$, g factors uniquely through $\ker(f)$:

$$\begin{tikzcd} Z \arrow[d, dashed] \arrow[dr, "g"] \arrow[drr, bend left, "0"] & & \\ K \arrow[r, "\ker(f)"'] & A \arrow[r, "f"'] & B \end{tikzcd}$$

When these exist, these are unique up to unique isomorphism, of course. Note that a kernel is a morphism, not an object! When we are not being careful, we will sometimes use the word kernel for the domain K.

Example: (Kernels in R-mod, $Ch(\mathcal{A})$, PreSh(X), Sh(X).)
The kernel of $\varphi : M \to N$ in R-mod is what you expect it to be, namely the inclusion map $\ker \varphi \to M$.

The kernel of a map of chain complexes $\varphi_* : B_* \to C_*$ is constructed as follows. For each i, let K_i be the domain of the kernel of the map $\varphi_i : B_i \to C_i$. We obtain a differential $d_i : K_i \to K_{i-1}$ as the unique map completing the following diagram:

$$\begin{tikzcd} K_i \arrow[r, "\ker \varphi_i"] \arrow[d, dashed] & B_i \arrow[r, "\varphi_i"] \arrow[d] & C_i \arrow[d] \\ K_{i-1} \arrow[r, "\ker \varphi_{i-1}"] & B_{i-1} \arrow[r, "\varphi_{i-1}"] & C_{i-1} \end{tikzcd}$$

The map $\ker \varphi_* : K_* \to B_*$ is the kernel of φ_*. (When $\mathcal{A}$ is R-mod, the differentials of K_* are just the restrictions of B_*'s.)

The kernel of a map of presheaves is constructed in a similar way. Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves. For each U, we define $\mathscr{K}(U)$ to be the domain of the kernel of the map $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$. We define restriction maps $\mathscr{K}(U) \to \mathscr{K}(V)$ as the unique map completing the diagram:

$$\begin{tikzcd} \mathscr{K}(U) \arrow[r, "\ker \varphi_U"] \arrow[d, dashed] & \mathscr{F}(U) \arrow[r, "\varphi_U"] \arrow[d] & \mathscr{G}(U) \arrow[d] \\ \mathscr{K}(V) \arrow[r, "\ker \varphi_V"] & \mathscr{F}(V) \arrow[r, "\varphi_V"] & \mathscr{G}(V) \end{tikzcd}$$

The family of morphisms $\ker \varphi_U : \mathscr{K}(U) \to \mathscr{F}(U)$ then define the kernel of a morphism of presheaves.

Kernels of sheaves are defined in exactly the same way as kernels of presheaves.

The dual definition is that of a cokernel.

Definition. A **cokernel** of a map $f : A \to B$ is a morphism $\operatorname{coker}(f) : B \to C$ so that $\operatorname{coker}(f) \circ f = 0$ and given any map $g : B \to Z$ so that $g \circ f = 0$, g factors uniquely through $\operatorname{coker}(f)$:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[drr, bend right, "0"'] & B \arrow[r, "\operatorname{coker} f"] \arrow[dr, "g"] & C \arrow[d, dashed] \\ & & Z \end{tikzcd}$$

Example: (Cokernels in R-mod, $Ch(\mathcal{A})$, PreSh(X), Sh(X).) The cokernel of a map $\varphi : M \to N$ in R-**mod** is the quotient map $N \mapsto N/\varphi(M)$.

The cokernel of a map of chain complexes $\varphi : B_* \to C_*$ is done term-by-term, just as before.

The cokernel of a map of presheaves is done term-by-term, just as before.

The cokernel of a map of sheaves is a bit more complicated. (Picture of representatives that fail to glue in the presheaf cokernel of $2\pi\underline{\mathbb{Z}}(S^1) \to C(S^1)$.) In order to rectify the problem of sections not gluing, we add in sections using a process called **sheafification**. It turns out that sheafification defines an (exact) functor.

Definition. The **image** of $f : A \to B$ is the map $\ker(\operatorname{coker}(f)) : K \to B$. The **coimage** of $f : A \to B$ is the map $\operatorname{coker}(\ker(f)) : A \to C$.

Example: Images in R-mod.

Take any morphism $\varphi : M \to N$. The cokernel is $\pi : N \to N/\varphi(M)$, which has kernel the inclusion $\varphi(M) \to N$, so the image of $\varphi : M \to N$ is inclusion $\varphi(M) \to N$.

For comparison, the kernel of φ is $\ker(\varphi) \to M$, which has cokernel $M \to M/\ker(\varphi) \cong \varphi(M)$, so the coimage of $\varphi : M \to N$ is the quotient map $M \to \varphi(M)$.

Roughly speaking, the categorical image is what the image should be if you think of images as subobjects of the codomain, while the coimage is what the image should be if you think of images as quotients of the domain.

4. Abelian Categories

Definition. An **abelian category** is an additive category so that

(i) Every map has a kernel and a cokernel.
(ii) For all morphisms f, the natural map $\operatorname{coim}(f) \to \operatorname{im}(f)$ is an isomorphism.

What is this natural morphism? (Derivation in a diagram.)

Theorem. Fix an abelian category $\mathcal{A}$. In this category,

(i) $0 \to A \to B$ is exact if and only if $A \to B$ is a monomorphism.
(ii) $A \to B \to 0$ is exact if and only if $A \to B$ is an epimorphism.
(iii) $0 \to A \to B \to 0$ is exact if and only if $A \to B$ is an isomorphism.

Proof. (i) The cokernel of $0 \to A$ is the identity map $A \xrightarrow{1_A} A$, which has as kernel $0 \to A$. So the image of $0 \to A$ is $0 \to A$. Thus $0 \to A \to B$ is exact if and only if $0 \to A$ is the kernel of $A \to B$.

So we are showing that being a monomorphism is the same as having kernel $0 \to A$. Suppose first that $A \xrightarrow{f} B$ has kernel $0 \to A$. Let g, h be two morphisms $Z \to A$ so that $f \circ g = f \circ h$. Then, by linearity, $f \circ (g - h) = 0$. By the universal property of kernels, $g - h$ factors through $0 \to A$, so $g - h$ must be the zero morphism. This implies $g = h$, so the defining property of monomorphisms is verified.

Conversely, suppose $A \xrightarrow{f} B$ is a monomorphism. Suppose $g : Z \to A$ is a morphism so that $f \circ g = 0$. Note that there is another map with this property, namely the zero morphism, $0 : Z \to A$. Since f is a monomorphism, $g = 0$. It follows that g factors uniquely through $0 \to A$. Since g was arbitrary, this verifies that $0 \to A$ has the universal property of the kernel.

(Note: So far, we have only used that $\mathcal{A}$ is additive and kernels and cokernels exist.)

(ii) Duality.

(iii) Suppose first that $A \to B$ is an isomorphism. Then $A \to B$ is both monic and epic, so the sequence is exact by parts (i) and (ii).

Suppose conversely that $0 \to A \to B \to 0$ is exact. Then $B \to 0$ is the cokernel of $A \to B$ and $0 \to A$ is the kernel of $A \to B$. This makes the image of $A \to B$ the morphism $\mathrm{id}_B : B \to B$ and the coimage of $A \to B$ the morphism $\mathrm{id}_A : A \to A$. By definition of abelian category, the natural map from the coimage, A, to the image, B, is an isomorphism. The following diagram commutes:

$$\begin{tikzcd} & A \arrow[r, "\sim"] \arrow[d, equal] & B \arrow[d, equal] & \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

So the map $A \to B$ is an isomorphism.

□

Example: R-mod, Ab, Ch($\mathcal{A}$) ($\mathcal{A}$ is abelian), Presheaves, Sheaves are abelian categories.

Theorem. (Freyd-Mitchell Embedding Theorem) If $\mathcal{A}$ is a small abelian category, then there is an exact, fully faithful functor from $\mathcal{A}$ into $\mathbb{R}$-**mod**, which embeds $\mathcal{A}$ as a full subcategory in the sense that $\mathrm{Hom}_{\mathcal{A}}(M, N) \cong \mathrm{Hom}_R(M, N)$.