INTRO TO CATEGORIES: PART 3

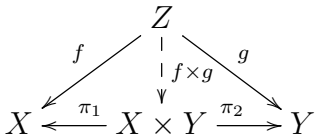
SEBASTIAN BOZLEE

The focus today will be on universal properties. These will be both a new means of definition and provide a new means of proof. Since universal properties are category theoretic, they have applications throughout mathematics. They are very strange when you first see them, so we'll start with an example, and I'll try to share some of the tricks I've found over time to interpret these in the context of that example.

1. Our First Universal Property: Products

We often have a notion of cartesian product of two objects. For example, given sets S and T, we may form the cartesian product $S \times T$, or given two groups, G and H, we may form a product group, $G \times H$. The categorical definition that captures this uses a universal property.

Definition. Given two objects X and Y in a category $\mathscr{C}$, a **product** of X and Y consists of an object $X \times Y$ and two morphisms $X \times Y \xrightarrow{\pi_1} X$ and $X \times Y \xrightarrow{\pi_2} Y$ such that the following **universal property** is satisfied: for any pair of maps $f : Z \to X$ and $g : Z \to Y$, there exists a unique morphism $f \times g : Z \to X \times Y$ so that the following diagram commutes:

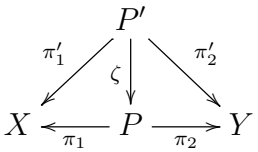


(**commutes** means that any chain of compositions between two points in the diagram are equal. Here, it means $\pi_1 \circ (f \times g) = f$ and $\pi_2 \circ (f \times g) = g$.)

(A universal property is a property that fits the pattern of "for all arrows such that ... there is a unique arrow such that everything commutes." There may be a formal definition, but I don't know it.)

Note that the product is not just the object $X \times Y$, but the object $X \times Y$ together with the two maps $X \times Y \to X$ and $X \times Y \to Y$.

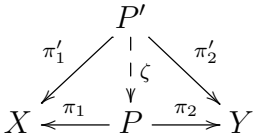
Theorem. (Products are "unique up to unique isomorphism.") Suppose that (P, π_1, π_2) and (P', π_1', π_2') are both products of X and Y. Then there is a unique isomorphism $\zeta : P' \to P$ so that



commutes.

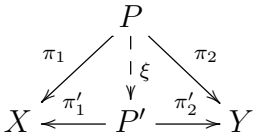
Date: June 8, 2017.

Proof. Suppose that (P, π_1, π_2) and (P', π_1', π_2') are both products of X and Y. By definition of P, since there are maps $\pi_1' : P' \to X$ and $\pi_2' : P' \to Y$, there is a unique map $\zeta : P' \to P$ so that the diagram



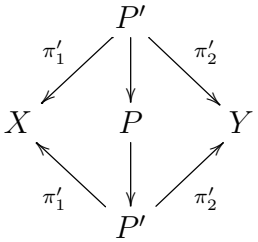
commutes.

Similarly, there is a unique map $\xi : P \to P'$ so that the diagram



commutes.

Composing the two maps, we get that the diagram



commutes. By the uniqueness in the definition of the product, the composition down the middle must be the identity on P'.

A symmetric argument shows that the composition the other way is the identity on P. Therefore the maps $\xi : P \to P'$ and $\zeta : P' \to P$ are isomorphisms. □

Example: (Products are products.)

Category	product
Set	cartesian product
Grp	product group
Ab	product group
$\mathbb{R}$-**Mod**	product vector space
Top	product topological space

We are omitting the projection maps. In each case, the projection map $\pi_1 : X \times Y \to X$ is defined by $(x, y) \mapsto x$ and $\pi_2 : X \times Y \to Y$ is defined by $(x, y) \mapsto y$. You may verify that given morphisms $f : Z \to X$ and $g : Z \to Y$ the unique map from Z to $X \times Y$ is defined by $z \mapsto (f(z), g(z))$.

Exercise 1. What are the products in a poset category? Must they exist?

Moral 1. Universal properties are isomorphisms of Hom sets.

For example, in the universal property of products, we are told that there is exactly one morphism $Z \to X \times Y$ for every two morphisms $Z \to X$ and $Z \to Y$. Conversely, given a

morphism from $Z \to X \times Y$, we can get back maps $Z \to X$ and $Z \to Y$ by composing with the projection maps. That is,

$$\mathrm{Hom}(Z, X \times Y) \cong \mathrm{Hom}(Z, X) \times \mathrm{Hom}(Z, Y).$$

Example: Here is a slick fake proof that the categorical product in **Set** is the cartesian product. Suppose a product (P, π_1, π_2) exists in **Set** of S and T. Then note that

$$P \cong \mathrm{Hom}(*, P) \cong \mathrm{Hom}(*, S) \times \mathrm{Hom}(*, T) \cong S \times T.$$

The second isomorphism comes from the universal property. (Why is this a fake proof?)

Key to our fake proof was considering only the maps from a point. This gives us another moral.

Moral 2. Universal properties are sometimes better understood for restricted classes of maps.

Sometimes you will get a better understanding of why a universal property works by first thinking of why it works for maps from a particular object, or monomorphisms, or epimorphisms.

Moral 3. Universal properties are constructions of maps.

Example: We said above that in **Set**, **Grp**, **Ab**, $\mathbb{R}$-**mod**, and **Top** that given morphisms $f : Z \to X$ and $g : Z \to Y$ the unique map from Z to $X \times Y$ is defined by $z \mapsto (f(z), g(z))$.

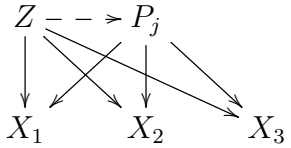
In other words, knowing how to get from the maps provided to the unique map gives a method of constructing maps. This may not be very impressive for **Set**, say, but for **Top** there is the implicit assertion that $z \mapsto (f(z), g(z))$ is continuous. Knowing that the construction yields a morphism can save you some work in verifying properties such as continuity or being a group homomorphism. These constructions are usually hidden in the proof that such-and-such has a universal property, but despite being hidden are used all the time.

Moral 4. Universal properties give an easy way to prove that objects are isomorphic.

Theorem. (Products are "associative") Denote by $X \times Y$ any product of X and Y. Given three objects X_1, X_2, X_3 in a category such that $P_1 = (X_1 \times X_2) \times X_3$ and $P_2 = X_1 \times (X_2 \times X_3)$ exist. Then $(X_1 \times X_2) \times X_3 \cong X_1 \times (X_2 \times X_3)$.

Proof. Define projection maps $\pi_{i,j} : P_j \to X_i$ in the obvious way.

We will verify that both have the universal property that given any three maps $f_i : Z \to X_i$ for $i = 1, 2, 3$, there is a unique map $Z \to P_j$ so that the diagram



commutes. By an argument similar to the one that showed products are unique up to isomorphism, this will give an isomorphism between P_1 and P_2.

TO DO: Not typed up because of the diagrams. □

Using this idea, we can define products for any set of objects.

Definition. The product of a set of objects $\{X_\lambda\}_{\lambda\in\Lambda}$ in a category $\mathscr{C}$ is an object P together with morphisms $\pi_\lambda : P \to X_\lambda$ for each $\lambda \in \Lambda$ such that for all systems of maps $\{f_\lambda : Z \to X_\lambda\}_{\lambda\in\Lambda}$, there is a unique map $f : Z \to P$ so that

$$f_\lambda = \pi_\lambda \circ f$$

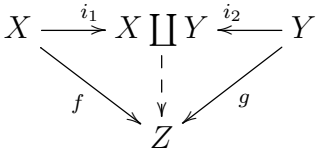
for all λ.

2. Coproducts

We said a week ago that flipping all of the arrows of any good categorical definition gives another good categorical definition. What we get when we flip the arrows on a product is a coproduct:

Definition. Given two objects X and Y in a category $\mathscr{C}$, an object $X \coprod Y$ together with maps $i_1 : X \to X \coprod Y$ and $i_2 : Y \to X \coprod Y$ is a **coproduct** of X and Y if the following universal property holds:

For all pairs of maps $f : X \to Z$ and $g : Y \to Z$, there exists a unique map $f \coprod g : X \coprod Y \to Z$ so that the diagram



commutes.

As before, $X \coprod Y$ is unique up to unique isomorphism. This time, we end up with the "right" notion of a "disjoint union."

Example:

Category	coproduct	$i_1 : X \to X \coprod Y$
Set	disjoint union	$x \mapsto x$
Grp	free product	$g \mapsto g$
Ab	direct sum	$g \mapsto (g, 0)$
$\mathbb{R}$-**Mod**	direct sum	$v \mapsto (v, 0)$
Top	disjoint union	$x \mapsto x$

Notice that in the universal property of coproducts, the arrows go "out" rather than "in." Universal properties roughly split into these two kinds. Loosely speaking, objects with arrows-in universal properties are called "limits," and objects with arrows-out universal properties are called "colimits." Examples of limits include products, kernels, intersections, and preimages. Examples of colimits include coproducts, quotients, and unions.