INTRO TO CATEGORIES: PART 2

SEBASTIAN BOZLEE

1. More on Functors

Definition. A **contravariant functor** F from a category $\mathscr{C}$ to a category $\mathscr{D}$ is defined in the same way as a functor except that arrows switch direction. F associates to each object A of $\mathscr{C}$ an object $F(A)$ of $\mathscr{D}$ and to each morphism $f : A \to B$ a morphism $F(f) : F(B) \to F(A)$ subject to the constraints that

(i) $F(1_A) = 1_{F(A)}$ for all $a \in A$.
(ii) $F(f \circ g) = F(g) \circ F(f)$ for every pair of composable morphisms $f : A \to B$ and $g : B \to C$ in $\mathscr{C}$.

When you want to emphasize that a functor is not a contravariant functor, you call it a **covariant** functor.

Example: (Hom functor) Recall that if A, B are objects of a category $\mathscr{C}$, then $\mathrm{Hom}_{\mathscr{C}}(A, B)$ denotes the set of all morphisms from A to B. We will omit the $\mathscr{C}$ when the category is clear from context.

Let $\mathscr{C}$ be any category and A any object of $\mathscr{C}$. Then $\mathrm{Hom}(A, -)$ defines a covariant functor and $\mathrm{Hom}(-, A)$ defines a contravariant functor from $\mathscr{C}$ to **Set**.

In more detail, the functor $\mathrm{Hom}(A, -)$ is defined on objects by $X \mapsto \mathrm{Hom}(A, X)$ and on morphisms by

$$(f : X \to Y) \mapsto f^* : \mathrm{Hom}(A, X) \to \mathrm{Hom}(A, Y)$$
$$\phi \mapsto f \circ \phi$$

I can only remember this by thinking of the following diagram:

$$\begin{tikzcd} A \arrow[d, "\phi"'] \arrow[dr, "f\circ\phi"] & \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Similarly, we define $\mathrm{Hom}(-, A)$ on objects by $X \mapsto \mathrm{Hom}(A, X)$ and on morphisms by

$$(f : X \to Y) \mapsto f^* : \mathrm{Hom}(Y, A) \to \mathrm{Hom}(X, A)$$
$$\phi \mapsto \phi \circ f$$

Here is the corresponding diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\psi\circ f"'] & Y \arrow[d, "\psi"] \\ & A \end{tikzcd}$$

Date: June 8, 2017.

These diagrams also help to remember which "slot" is covariant and which is contravariant.

2. Kinds of Morphisms

Last time we saw that injective functions told us we had a subset of a set, and injective group homomorphisms told us we had a subgroup of a group. If categories are to capture the information encoded in the existence of injective functions, then there better be a way to detect something like injectivity categorically. However, in a category, we do not have the right to think of our objects as sets, so we can't test for injectivity by plugging in elements to our functions. The next best thing we can do is to define the following:

Definition. A morphism $B \xrightarrow{f} C$ in $\mathscr{C}$ is a **monomorphism** if for any pair of morphisms $A \xrightarrow{g} B$ and $A \xrightarrow{h} B$, $f \circ g = f \circ h$ implies $g = h$.

The picture is $A \rightrightarrows B \xrightarrow{f} C$.

Example:

In **Set**, **Grp**, **Ab**, $\mathbb{R}$-**mod**, and **Top**, the monomorphisms are the injective morphisms. In a poset category, every morphism is a monomorphism. In **Cat**, the monomorphisms are the functors injective on both objects and morphisms.

Here's why injective functions coincide with monomorphisms in **Set**. Suppose that $f : S \to T$ is a function. Let $g_x : \{*\} \to S$ be defined by $* \mapsto x$ for each $x \in S$. Then $f \circ g_x = f \circ g_y$ if and only if $f(x) = f(y)$. (Comment: it is as if plugging in an element to a function is a special case of function composition. Hm.) Moreover $g_x = g_y$ if and only if $x = y$. So if f is a monomorphism, then $f(x) = f(y)$ implies $f \circ g_x = f \circ g_y$ implies $g_x = g_y$ implies $x = y$. So f is injective.

Conversely, if f is injective and $g, h : A \to S$ are functions so that $f \circ g = f \circ h$, then $f(g(a)) = f(h(a))$ for all $a \in A$, which implies $g(a) = h(a)$ for all a, which implies $g = h$.

Definition. A **subobject** of an object B in $\mathscr{C}$ is a monomorphism $A \to B$.

Notice that the function is part of the definition of a suboject. This is typical for categorical definitions.

The corresponding concept for surjective functions is the following:

Definition. A morphism $A \xrightarrow{f} B$ in $\mathscr{C}$ is an **epimorphism** if for any pair of morphisms $g, h : B \to C$, $g \circ f = h \circ f$ implies $g = h$.

The picture is $A \xrightarrow{f} B \rightrightarrows C$.

Example: In **Set**, **Grp**, **Ab**, and $\mathbb{R}$-**mod** epimorphisms coincide with surjective morphisms.

However, epimorphisms need not be surjective morphisms in all cases. For example, there is a category of commutative rings with unity, whose objects are rings and whose morphisms are ring homomorphisms - functions preserving the identity and multiplication. In this category, the inclusion map $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism.

Remark: Notice that the definition of epimorphism is the same as the definition of monomorphism with all the arrows reversed. This is an instance of the **principle of duality**. Informally, any good categorical concept, when arrows are reversed, yields another good categorical concept. The reason for this is that reversing the arrows in any category $\mathscr{C}$ yields another category $\mathscr{C}^{op}$, called the opposite category. If you take a good categorical concept, reverse its arrows and apply it to $\mathscr{C}$, you are really applying the original concept to the opposite category $\mathscr{C}^{op}$.

Definition. A morphism $A \xrightarrow{f} B$ in $\mathscr{C}$ is an **isomorphism** if there is another morphism $f^{-1} : B \to A$ so that $f \circ f^{-1} = 1_B$ and $f^{-1} \circ f = 1_A$.

Example:

Category	Isomorphisms
Set	bijections
Grp	group isomorphisms
Ab	group isomorphisms
$\mathbb{R}$-**Mod**	linear isomorphisms
Top	homeomorphisms

Note that isomorphisms are not just bijective morphisms: consider the identity map in **Top** from $\{1, 2\}$ to $\{1, 2\}$ where the domain is given the discrete topology and the codomain is given the indiscrete topology. Nor are they the morphisms which are both monic and epic: the same map gives an example.

3. Wrap up

Today we saw a little more about functors and we saw how properties that we usually thinking of in terms of elements of sets - things like injectivity and surjectivity - can bedefined in a uniform way across categories solely in terms of how functions compose.

Next time, we'll talk about universal properties.