INTRO TO CATEGORIES: PART 1

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1. Motivation

Category Theory is an organizing language for mathematics based on the central idea that **functions express relationships**, and therefore a useful way to reason about and define mathematical concepts is in terms of patterns of functions. Consider the following example.

Example: Let S and T be sets, and f a function between them.

If...	Then...
f is injective	$\lvert S\rvert \leq \lvert T\rvert$, S is "isomorphic" to the subset $f(S)$ of T
f is surjective	$\lvert S\rvert \geq \lvert T\rvert$, T is like S with a few elements identified, "quotient set of S"
f is bijective	$\lvert S\rvert = \lvert T\rvert$, S and T are "isomorphic sets."

So far so good. Something similar works for groups, only this time we should use group homomorphisms instead of arbitrary functions. Let G and H be groups and let $f : G \to H$ be a group homomorphism.

If...	Then...
f is injective	G is isomorphic to the subgroup of H, namely $f(G)$
f is surjective	H is isomorphic to a quotient group of G, namely $G/\ker(f)$
f is bijective	G and H are isomorphic groups.

Just knowing that a function of a certain kind exists tells us something, both in the example of sets and the example of groups. Even better, this knowledge tells us the same things about sets as they do about groups.

2. Categories

With this as our motivation, let us define a category. Informally, a category is a collection of objects and a collection of functions between them. Formally,

Definition. A **category** $\mathscr{C}$ consists of

(i) A class of **objects**, $\mathrm{Ob}(\mathscr{C})$.
(ii) For each pair of objects, A, B, a set $\mathrm{Hom}_{\mathscr{C}}(A, B)$.
An element $\phi \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ is called a **morphism** or **arrow**, and is usually written $\phi : A \to B$ or $A \xrightarrow{\phi} B$. The object A is called the **domain** of ϕ and the object B is called the **codomain** of ϕ.

Date: June 8, 2017.

(iii) For each pair of morphisms $\phi : A \to B$ and $\psi : B \to C$, a **composite morphism** $\psi \circ \phi : A \to B$.

subject to the conditions that:

(i) Composition of morphisms is associative: if $A \xrightarrow{\phi} B \xrightarrow{\psi} C \xrightarrow{\rho} D$, then $(\rho \circ \psi) \circ \phi = \rho \circ (\psi \circ \phi)$.

(ii) For each object B, there is a morphism $B \xrightarrow{1_B} B$ so that for each morphism $A \xrightarrow{\phi} B$, $1_B \circ \phi = \phi$ and for each morphism $B \xrightarrow{\psi} C$, $\psi \circ 1_B = \psi$.

The word "class" here means a collection, just like a set, except it may be "too large" to behave in all the ways that we expect a set to behave. Unless you are curious about careful axiomatic set theory, you should regard this as a technicality.

Example: (Concrete Categories)

Category	Objects	Morphisms
Set	Sets	functions
Grp	Groups	group homomorphisms
Ab	Abelian groups	group homomorphisms
$\mathbb{R}$-**Mod**	Real vector spaces	linear maps
Top	Topological spaces	continuous maps

Composition is in each case function composition.

It's good to think about why each of these satisfies the axioms of a category. For definiteness, let's check the category of groups:

We have objects and morphisms, so the first two parts of the definition work out no problem. The composition of two group homomorphisms is again a group homomorphism, so the composition is well-defined. Function composition is always associative. Finally, we have identity morphisms, namely the functions $\mathrm{id}_G : G \to G$ taking $g \mapsto g$. Clearly, each id_G is a group homomorphism, and they compose in the way that the definition wants them to.

The category **Set** shows why we need to talk about classes, rather than sets: there is no set of all sets.

Example: (Categories don't need morphisms to actually be functions.)

Category	Objects	Morphisms	Composition
Poset category of $(P, \leq)$	Elements of P	if $A \leq B$, a unique morphism $A \to B$	the unique arrow
Group category of G	One object, $*$	$* \xrightarrow{g} *$ for each element $g \in G$	$(* \xrightarrow{g} *) \circ (* \xrightarrow{h} *) = * \xrightarrow{gh} *$

It's really interesting to think about why these are categories. Let's check the axioms for a poset category:

We have objects and morphisms, so those parts of the definition are clearly okay. Think about composition: if we have morphisms $A \to B$ and $B \to C$, this means that $A \leq B$ and $B \leq C$. Having a composite morphism $A \to C$ means that $A \leq C$. So the reason that our definition of composition works is that partial orders are transitive! Next, associativity holds trivially: since there is a unique morphism $A \to C$, any two morphisms $A \to C$ are equal. Finally, we need identity morphisms $A \to A$ for each object A. In other words, we need

$A \leq A$ for all $A \in P$. The reason that we have identity morphisms is that partial orders are reflexive.

Exercise 1. Prove that the group category of a group G is a category. Are there any group axioms that you don't need?

3. Functors

In the spirit of category theory, we should not only study categories but "morphisms of categories." A functor is the right idea.

Definition. A **functor** F from a category $\mathscr{C}$ to a category $\mathscr{D}$ is a mapping that associates to each object A of $\mathscr{C}$ an object $F(A)$ of $\mathscr{D}$ and that associates to each morphism $f : A \to B$ in $\mathscr{C}$ a morphism $F(f) : F(A) \to F(B)$ in $\mathscr{D}$, subject to the constraints that

(i) $F(1_A) = 1_{F(A)}$ for every A in $\mathscr{C}$, and
(ii) $F(f \circ g) = F(f) \circ F(g)$ for every pair of composable morphisms $f : A \to B$ and $g : B \to C$ in $\mathscr{C}$.

Example: (Forgetful functors) We can often get a functor just by "forgetting" some structure of our objects. For example, there are functors from **Grp**, **Ab**, $\mathbb{R}$-**mod**, and **Top** to **Set** that take each object to its underlying set and each morphism to itself.

Example: (Free group functor) Given a set S, we can form the **free group generated by** S. This is the group $F(S)$ that has underlying set consisting of all words in the symbols s and s^{-1} for each $s \in S$, without any adjacent symbols s and s^{-1}. The multiplication is defined by concatenating strings and removing any adjacent symbols s and s^{-1} that result.

Given a function $f : S \to T$ in **Set**, we get a morphism $Ff : F(S) \to F(T)$ by taking each symbol s in a word to the symbol $f(s)$ and each symbol s^{-1} to the symbol $f(s)^{-1}$.

As a general moral, we should expect that if we have a systematically defined way to match objects between categories then that matching can naturally be extended to a functor by defining what it does to morphisms.

If we have two functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{E}$, then there is a functor $F \circ G : \mathscr{C} \to \mathscr{D}$ defined in the obvious way. This gives us a category of categories! For technical reasons, we need to restrict to **small categories**: categories whose class of objects is actually a set. With this in mind we can add to our list of categories:

Category	Objects	Morphisms
Cat	Small categories	functors

4. Wrap up

Today we've defined a category, which was an abstract definition capturing "things and structure preserving maps between them." We next defined a notion of functor, which was either "a structure preserving map between categories" or a "systematic assignment of objects of one category to objects of another category."

Next time, we'll talk a little bit more about functors, then give categorical ways of defining things like subobjects, quotients, and isomorphisms.