INJECTIVE MODULES ARE "ALGEBRAICALLY CLOSED"

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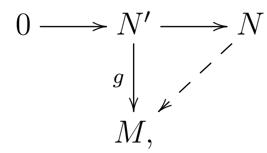
Let R be a commutative ring with unity.

Definition 1. An R-module M is **injective** if, for any injective R-module homomorphism $f : N' \to N$ and homomorphism $g : N' \to M$ there exists a function $\tilde{g} : N \to M$ so that $\tilde{g} \circ f = g$. That is, every solid diagram



where the row is exact, admits a completion.

Injective modules play an important theoretical role in homological algebra and algebraic geometry. However, the definition is somewhat abstract. In this note, we will provide a reinterpretation of injectivity that may feel more concrete.

An algebraically closed field is one in which every non-constant polynomial equation in one variable has a root. We'd like to define an analogous notion for R-modules, which will turn out to be equivalent to the notion of injective R-module. We should start by finding an analogous kind of equation to ask for solutions to.

When working with fields the operations we have available to us are multiplication and addition. Polynomials are precisely what pop out when you use these operations to combine field elements and variables.

When working with R-modules, on the other hand, the operations we have available are scalar multiplication and addition. The kinds of expressions that pop out when combining these are the R-linear combinations of variables and module elements. Since an R-linear combination of module elements is equal to another module element, any equation involving R-linear combinations of variables and elements can be reduced to the form

$$r_1 x_1 + r_2 x_2 + \cdots + r_k x_k = m$$

where the rs are ring elements, the xs are variables, and m is a module element. These are the equations we will be interested in solving.

Although we only consider single-variable polynomials when defining an algebraically closed field, here we will keep multi-variable equations. (However, see Lemma 8.) Our experience with linear algebra tells us that solving a single equation of this kind is not very interesting, so we should ask for solutions to systems of them. In fact, we will consider infinitely many equations in infinitely many variables (with only finitely many variables in an equation).

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Finally, we cannot ask that every system of equations has a solution. For example, if all the rs in the equation above are zero and m is nonzero, it would be ridiculous to ask for a solution. (In fact, there is a similar problem with polynomial equations over fields. This is why we only consider non-constant polynomials in the definition of algebraically closed field.) So we will need a restriction on the types of systems of equations we consider, which we will call consistency.

We are now ready to state the definition.

Definition 2. Let M be an R-module. For any indexed set of indeterminates $\{x_i\}_{i\in I}$, a set of equations $\{r_1^{(j)}x_1^{(j)} + r_2^{(j)}x_2^{(j)} + \cdots + r_{k_j}^{(j)}x_{k_j}^{(j)} = m_j\}_{j\in J}$, where:

(1) each $r_i^{(j)} \in R$,
(2) each $x_i^{(j)}$ is one of the x_ks, $k \in I$,
(3) and each $m_j \in M$,

is called **consistent** if there is no R-linear combination of the equations resulting in 0 on the left hand side and a non-zero element of M on the right.

An R-module M is **algebraically closed** if any consistent set of equations as above admits a solution in M, that is, there is a function $f : \{x_i\}_{i\in I} \to M$ so that for all j in J,

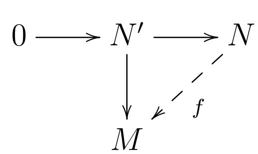
$$r_1^{(j)} f\left(x_1^{(j)}\right) + r_2^{(j)} f\left(x_2^{(j)}\right) + \cdots + r_{k_j}^{(j)} f\left(x_{k_j}^{(j)}\right) = m_j.$$

Proposition 3. An R-module M is injective if and only if M is algebraically closed.

Proof. ($\Longrightarrow$) Suppose first that M is injective. Let $X = \{x_i\}_{i\in I}$ be a set of indeterminates and $\{r_1^{(j)}x_1^{(j)} + r_2^{(j)}x_2^{(j)} + \cdots + r_{k_j}^{(j)}x_{k_j}^{(j)} = m_j\}_{j\in J}$ be a consistent set of equations in M with variables in X.

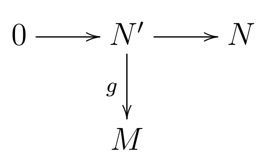
Let N' be the submodule of M generated by the m_js. Consider $N = (N' \oplus R^{\oplus X})/L$ where L is the submodule generated by the elements $r_1^{(j)}x_1^{(j)} + r_2^{(j)}x_2^{(j)} + \cdots + r_{k_j}^{(j)}x_{k_j}^{(j)} - m_j$. The natural map $N' \to N$ is injective since our set of equations was chosen to be consistent.

Since M is injective, there is an extension of the inclusion $N' \to M$ to N:



Then by the universal property of quotient modules, the restriction of f to the x_is gives a solution to the set of equations.

($\Longleftarrow$) Suppose conversely that M is algebraically closed. Consider a diagram



We want to find an extension of g to N. Note that we can factor g as

$$\begin{array}{ccccc} 0 & \longrightarrow & N' & \longrightarrow & N \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & N'/\ker g & \longrightarrow & N/\ker g \\ & & \downarrow{\scriptstyle \overline{g}} & & \\ & & M & & \end{array}$$

and that to extend g to N it is sufficient to extend $\overline{g}$ to $N/\ker g$. Therefore it is enough to consider the case where $g : N' \to M$ is injective. For notational convenience, we may as well take N' to be a submodule of M and g to be the inclusion.

Choose elements $X = \{x_i\}_{i \in I}$ of N so that X and N' jointly generate N (i.e. $RX + N' = N$.) Define a function $N' \oplus R^{\oplus I} \to N$ by the identity on N' and by $i \mapsto x_i$ on I. Let L be the kernel of this map, and note that each element l of L can be written in the form $r_1^{(l)} x_1^{(l)} + r_2^{(l)} x_2^{(l)} + \cdots + r_{k_l}^{(l)} x_{k_l}^{(l)} - m_l$ where each $r_i^{(l)} \in R$, each $x_i^{(l)} \in I$, and each $m_l \in N'$. This gives us a corresponding set of equations $\{r_1^{(l)} x_1^{(l)} + r_2^{(l)} x_2^{(l)} + \cdots + r_{k_l}^{(l)} x_{k_l}^{(l)} = m_l\}_{l \in L}$, consistent since $N' \to N$ is injective.

Now, since this set of equations is consistent and M is algebraically closed, we can find a function $f : X \to M$ satisfying them. By the universal property of quotients, this f induces a homomorphism of R-modules $\tilde{f} : N \to M$ making

$$\begin{array}{ccccc} 0 & \longrightarrow & N' & \longrightarrow & N \\ & & \downarrow{\scriptstyle g} & \swarrow{\scriptstyle \tilde{f}} & \\ & & M & & \end{array}$$

commute, as required. ☐

I like this characterization since it explains why injective modules are "large" and "hard to write down concretely," since the same things are true of algebraically closed fields. It suggests there should be an "algebraic closure," and it is shown elsewhere that there is one, called the injective hull. Furthermore, it gives an interpretation of finding an injective resolution that feels dual to the usual method of finding a free resolution: we introduce solutions to equations, then solutions to equations between the solutions, and so on.

The following standard facts on injective modules are easy to prove from this perspective.

Corollary 4. Injective modules are divisible.

Proof. Consider the equations $rx = m$. ☐

Corollary 5. Products of injective modules are injective modules.

Proof. Find solutions coordinate-wise. ☐

Corollary 6. If $\phi : R \to S$ is a surjective morphism of rings and M is an S-module so that M is injective as an R-module, then M is injective as an S-module.

Proof. By definition of restriction of scalars, for any $r_1, \ldots, r_n \in R$, $x_1, \ldots, x_n \in M$,

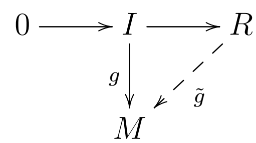
$$r_1 x_1 + \cdots + r_n x_n = \phi(r_1) x_1 + \cdots + \phi(r_n) x_n$$

So any equation with coefficients in S is equivalent to an equation with coefficients in R. Any lift of a consistent set of equations over S remains consistent over R, so we can find solutions to a consistent set of equations by lifting to R and solving. □

A warning: if M is injective as an S-module, it does not follow that M is injective as an R-module. For example, let $R = \mathbb{C}[a]_{(a)}$, and $M = S = R/a$. Then S is injective as an S-module, since every module is injective over a field. However, S is not injective as an R-module, since the equation $ax = 1$ does not admit a solution. The equation $ax = 1$ is consistent over R, but inconsistent over S, since ax reduces to 0.

Recall Baer's Lemma:

Lemma 7. (Baer's Lemma) An R-module M is injective if and only if for each ideal I of R and homomorphism of R-modules $g : I \to M$ there is an extension of g to R:



Translating this into our language of solving equations,

Lemma 8. (Baer's Lemma, equation version) An R-module M is injective if and only if any consistent set of equations of the form

$$\{r_j x = m_j\}_{j \in J}$$

admits a solution, where each $r_j \in R$, x is a variable, and each $m_j \in M$. We call such a set of equations a **single-variable** consistent set of equations.

Proof. The forward implication is clear by Proposition 3.

Suppose M has solutions to single-variable consistent sets of equations. We will use Baer's criterion to show M is injective. Suppose I is an ideal of R and $g : I \to M$ is an R-module homomorphism. Choose a system of generators $\{r_j\}_{j \in J}$ of I and set $m_j = g(r_j)$ for each $j \in J$. Consider the set of equations $\{r_j x = m_j\}_{j \in J}$. If there is an R-linear combination $\sum_{j \in J} s_j r_j$ of the r_js so that $\sum_{j \in J} s_j r_j = 0$, then

$$\sum_{j \in J} s_j m_j = \sum_{j \in J} s_j g(r_j) = g\left(\sum_{j \in J} s_j r_j\right) = 0,$$

so our set of equations is consistent. By hypothesis, there is an element $x \in M$ so that $r_j x = m_j$ for all $j \in J$. Then the map $R \to M$ taking $1 \mapsto x$ is the required extension of g. □

Examining the proof of the lemma, it is even enough to check only the equations corresponding to one set of generators, $\{r_i\}$, per ideal.

Corollary 9. If R is Noetherian, an R-module M is algebraically closed if and only if each **finite** consistent single-variable set of equations has a solution in M.

Proof. In the proof above, the set of generators $\{r_j\}$ of I can always be chosen to be finite. □

Example 10. Let $R = \mathbb{C}[\epsilon]/\epsilon^2$, $M = R$. Let us show that M is injective.

R has only three ideals, 0, (ϵ), and R. The equations associated to 0 and R are trivial to solve, so we're left with figuring out what the consistent equations of the form $\epsilon x = m$ are, then checking that they have solutions.

First, note that $\epsilon x = m \implies \epsilon m = \epsilon^2 x = 0$. So, in order for $\epsilon x = m$ to be consistent, m must be in the annihilator of ϵ. That is, $m \in (\epsilon)$. Therefore the equations we have to solve are those of the form $\epsilon x = \epsilon a$, where $a \in \mathbb{C}$. These have the obvious solution $x = a$, and we are done.

Example 11. Let $R = \mathbb{C}[a]_{(a)}$, $M = R/a = \mathbb{C}$. Let's compute the "smallest" injective module containing M.

Since R is a DVR, the only ideals of R are the trivial ideals and the ideals (a^n). This means we need to figure out what the consistent equations of the form $a^n x = m$ are, then introduce solutions to them.

On the former point, a single equation $rx = m$ is only inconsistent if there is some $s \in R$ so that $sr = 0$, but $sm \neq 0$. Our R is a domain, so this does not happen.

For $m = 0$, $a^n x = m$ has the trivial solution $x = 0$. On the other hand, $a^n x = 0$ for any $x \in M$, so $a^n x = m$ does not have a solution in M when m is nonzero. We'll have to introduce solutions.

Let $E = \mathbb{C}(a)/aC[a]_{(a)}$. M includes into E in a natural way, and there are solutions to the equations $a^n x = m$ for any $m \in E$. Moreover, no proper submodule has these properties. Therefore E is a "smallest" injective module containing M.

We get another pleasant proof of a basic result on injective modules.

Corollary 12. If R is a Noetherian ring, direct sums of injective modules are injective.

Proof. Let $\{M_i\}_{i\in I}$ be a set of injective R modules. To verify that $M = \bigoplus_{i\in I} M_i$ is injective, it is enough by the previous corollary to check that any finite, consistent single-variable set of equations in M admits a solution.

So let $\{r_j x = m_j\}_{j\in J}$ be a consistent set of equations where J is a finite indexing set. Only finitely many coordinates of the r_j and m_j are nonzero, so the whole set of equations is nonzero in only finitely many coordinates. We can use that the M_i are algebraically closed to find solutions in those coordinates, then combine these solutions to get a solution in M, that is, one with only finitely many nonzero coordinates. □