Midterm II

Intro to Discrete Math

MATH 2001

Spring 2022

Friday March 18, 2022

NAME: ______________________________

PRACTICE EXAM

SOLUTIONS

Question:	**1**	**2**	**3**	**4**	**5**	Total
Points:	20	20	20	20	20	100
Score:						

- The exam is closed book. You **may not use any resources** whatsoever, other than paper, pencil, and pen, to complete this exam.
- You **may not discuss the exam** with anyone except me, in any way, under any circumstances.
- You **must explain your answers**, and you will be **graded on the clarity of your solutions**.
- You must upload your exam as a single **.pdf** to **Canvas**, with the questions in the correct order, etc.
- You have 45 minutes to complete the exam. **We will spend the last 5 minutes of class to upload your exam to Canvas.**

1. (20 points) • **TRUE** or **FALSE**:

If $n \in \mathbb{N}$, then $\binom{2n}{n}$ is even.

If true, give a *direct proof* of the statement. If false, provide a *counter example*, and prove that it is a counter example. Your solution must start with the sentence, "*This statement is TRUE*," or the sentence, "*This statement is FALSE*."

SOLUTION:

Solution. This statement is TRUE. Indeed, we have

$$\begin{aligned}\binom{2n}{n} &= \frac{2n!}{n!(2n-n)!} = \frac{2n!}{n!n!} = \frac{(2n)(2n-1)(2n-2)\cdots(n+2)(n+1)}{n(n-1)(n-2)\cdots 2\cdot 1} \\ &= \frac{2n}{n}\cdot\frac{(2n-1)(2n-2)\cdots(n+2)(n+1)}{(n-1)(n-2)\cdots 2\cdot 1} \\ &= 2\cdot\binom{2n-1}{n},\end{aligned}$$

which is even, since $\binom{2n-1}{n}$ is an integer. □

1
20 points

2. (20 points) • In class we showed that the equation $x^2 + y^2 = 3$ has no rational solutions. Use this fact to give a *proof by contradiction* of the statement:

If k is an odd positive integer, then the equation $x^2 + y^2 = 3^k$ has no rational solutions.

SOLUTION:

Solution. Let k be an odd positive integer, and suppose for the sake of contradiction that there exists a rational solution $(x_0, y_0) \in \mathbb{Q}^2$ to the equation $x^2 + y^2 = 3^k$. In other words, we assume that there exist rational numbers x_0, y_0 such that

$$x_0^2 + y_0^2 = 3^k.$$

Since k is odd, we have that $k = 2n + 1$ for some non-negative integer n, and if we divide both sides of the equation above by 3^{2n}, we obtain the equation

$$\left(\frac{x_0}{3^n}\right)^2 + \left(\frac{y_0}{3^n}\right)^2 = 3.$$

Since $x_0/3^n$ and $y_0/3^n$ are rational numbers, the equation above would say that $x_0/3^n$ and $y_0/3^n$ give a rational solution to the equation $x^2 + y^2 = 3$, which we know is impossible. Consequently, since we have arrived at a contradiction, our assumption that there exists a rational solution $(x_0, y_0) \in \mathbb{Q}^2$ to the equation $x^2 + y^2 = 3^k$ was false.

Therefore, if k is an odd positive integer, then the equation $x^2 + y^2 = 3^k$ has no rational solutions. □

2
20 points

3. (20 points) • For all real numbers $a, b \in \mathbb{R}$, give a *proof by induction* that for each natural number n the following statement is true:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

You may use, *without proof*, the fact that $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$.

SOLUTION:

Solution. For each natural number n we have the statement:

$$p(n): \quad (a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

We start with the case $n = 1$, and we check that $p(1)$ is true:

$$\sum_{k=0}^{1} \binom{1}{k} a^k b^{1-k} = b + a = (a+b)^1.$$

We now perform the inductive step. We assume that $p(m)$ is true for all $m \leq n$ for some natural number $n \geq 1$. In other words, we assume that $(a+b)^m = \sum_{k=0}^{m} \binom{m}{k} a^k b^{m-k}$ for all $m \leq n$ for some natural number $n \geq 1$. We then use this to show that $p(n+1)$ is true:

$$(a+b)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^k b^{n+1-k}.$$

Here is the computation:

$$(a+b)^{n+1} = (a+b)^n (a+b) = \left(\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \right) (a+b) = \left(\sum_{k=0}^{n} \binom{n}{k} a^{k+1} b^{n-k} \right) + \left(\sum_{k=0}^{n} \binom{n}{k} a^k b^{n+1-k} \right)$$

$$= \binom{n}{0} b^{n+1} + \sum_{k=1}^{n} \left(\binom{n}{k-1} + \binom{n}{k} \right) a^k b^{n+1-k} + \binom{n}{n} a^{n+1}$$

$$= \binom{n+1}{0} b^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} a^k b^{n+1-k} + \binom{n+1}{n+1} a^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^k b^{n+1-k},$$

where, for the second equality, we are using that $p(n)$ is true. This completes the proof. □

Although this is not asked for in the problem, here is a proof of the fact that $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$:

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(n-k+1)!(k-1)!} + \frac{n!}{(n-k)!k!} = \frac{n!k}{(n-k+1)!k!} + \frac{n!(n-k+1)}{(n-k+1)!k!}$$

$$= \frac{n!(k+n-k+1)}{(n+1-k)!k!} = \frac{(n+1)!}{(n+1-k)!k!} = \binom{n+1}{k}.$$

3
20 points

4. (20 points) • Suppose R is an equivalence relation on a set A, with four equivalence classes. *How many different equivalence relations S on A are there for which $R \subseteq S$?* You must prove that your answer is correct.

SOLUTION:

Solution. There are 15 different equivalence relations S on A for which $R \subseteq S$.

To see this, we will convert the question into a statement about partitions. Recall that, as an exercise given in class on Monday March 14, we were asked to prove:

If R and S are equivalence relations on a set A, then $R \subseteq S$ if and only if for all $X \in P_R$ there exists $Y \in P_S$ with $X \subseteq Y$.

Here we are using the notation $P_R = A/R$ for the partition of A associated to the equivalence relation R, which is by definition the set of equivalence classes of R, and similarly for S.

This means that $R \subseteq S$ if and only if the partition P_S is obtained from the partition P_R by taking unions of sets in P_R. In our case we have an equivalence relation R on A that leads to a partition of A with four nonempty sets, say

$$A = A_1 \sqcup A_2 \sqcup A_3 \sqcup A_4.$$

Each equivalence relation S on A with $R \subseteq S$ corresponds to a partition of A obtained by taking unions of some of the A_i. For instance, we could take the partition of A into the three sets $(A_1 \cup A_2)$, A_3, and A_4. In total, all of the combinations give us

$$1 + \binom{4}{2} + \frac{1}{2}\binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 1 + 6 + 3 + 4 + 1 = 15$$

distinct partitions. Indeed, we could simply leave the partition alone, keeping the partition with the four sets A_1, A_2, A_3, and A_4. Or we could choose any two sets in the partition, and take their union; this gives us $\binom{4}{2}$ partitions with three sets. For instance, we could choose A_1 and A_2, and then arrive at the partition of A into the three sets $(A_1 \cup A_2)$, A_3, and A_4. Alternatively, we could choose any two sets in the partition, and take their union, and then also take the union of the other two sets; this gives us $\frac{1}{2}\binom{4}{2}$ partitions with two sets. For instance, we could choose A_1 and A_2, and then arrive at the partition of A into the two sets $(A_1 \cup A_2)$ and $(A_3 \cup A_4)$. Note that this is the same partition we would obtain if we had chosen A_3 and A_4. We can also choose any three sets and take their union; this gives us $\binom{4}{3}$ partitions with two sets. For instance, we could choose A_1, A_2, and A_3, and arrive at the partition of A into the two sets $(A_1 \cup A_2 \cup A_3)$ and A_4. Or, finally, we could choose all four sets and take their union, to gives us $\binom{4}{4}$ partitions with one set; i.e., the partition of A into the one set $(A_1 \cup A_2 \cup A_3 \cup A_4)$. $\square$

Another approach to this problem is to show that given a set A and an equivalence relation R on A, then equivalence relations S on A with $R \subseteq S$ correspond to equivalence relations on A/R. Since in our problem A/R has four elements, we are then asking for the number of equivalence relations on a set with four elements. The argument above shows that there are 15 equivalence relations on a set with four elements.

Here is a solution to the exercise mentioned above, to show that:

If R and S are equivalence relations on a set A, then $R \subseteq S$ if and only if for all $X \in P_R$ there exists $Y \in P_S$ with $X \subseteq Y$.

First assume that $R \subseteq S$, and let $X \in P_R$. Then by definition, there exists $a \in A$ such that X is the equivalence class of a for R; i.e., $X = [a]_R = \{x \in A : x \sim_R a\} = \{x \in A : (x,a) \in R\}$. At the same time, we have

$$X = [a]_R = \{x \in A : x \sim_R a\} = \{x \in A : (x,a) \in R\} \subseteq \{x \in A : (x,a) \in S\} = [x]_S \in P_S,$$

completing the proof that there exists $Y \in P_S$ with $X \subseteq Y$.

Conversely, assume that for all $X \in P_R$ there exists $Y \in P_S$ with $X \subseteq Y$. We want to show that $R \subseteq S$. So let $(a,b) \in R$. Then we have $[a]_R \in P_R$, and we have by assumption $Y \in P_S$ such that $[a]_R \subseteq Y$. Since $a \sim_R b$, we have $a,b \in [a]_R \subseteq Y$, so that $a,b \in Y$. By definition, there exists $c \in A$ such that $[Y] = [c]_S = \{x \in A : x \sim_S c\}$. Thus we have $a \sim_S c$ and $b \sim_S c$, so that $a \sim_S b$, by symmetry and transitivity. In other words, $(a,b) \in S$, completing the proof.

4
20 points

5. • **TRUE** or **FALSE**. For this problem, and this problem only, **you do not need to justify your answer**.

(a) (4 points) **TRUE** or **FALSE** (circle one). The LaTeX code

```
x^100+3\pi x^2+5
```

produces the following:

$$x^{100} + 3\pi x^2 + 5$$

SOLUTION: FALSE. It produces: $x^100 + 3\pi x^2 + 5$. One needs to use

```
x^{100}
```

(b) (4 points) **TRUE** or **FALSE** (circle one). If R and S are equivalence relations on a set A, then $R \cap S$ is also an equivalence relation on A.

SOLUTION: TRUE. I leave this to you as an exercise to check that $R \cap S$ is reflexive, symmetric, and transitive.

(c) (4 points) **TRUE** or **FALSE** (circle one). The empty set defines a reflexive relation on any set.

SOLUTION: FALSE. If the set A is not the empty set, then the empty set $\emptyset \subseteq A \times A$ is not reflexive since for any $a \in A$, we have $(a, a) \notin \emptyset$.

(d) (4 points) **TRUE** or **FALSE** (circle one). If $\sim$ is an equivalence relation on a set A and $a \in A$, then the equivalence class of a is the set $[a] = \{x \in A : \exists y \in A, x \sim y\}$.

SOLUTION: FALSE. $[a] = \{x \in A : x \sim a\}$, while $\{x \in A : \exists y \in A, x \sim y\} = A$. For a counter example to the statement, i.e., an example where $[a] \neq \{x \in A : \exists y \in A, x \sim y\} = A$, take equivalence modulo 2 on the integers, and note that $[0] = \{2n : n \in \mathbb{Z}\} \neq \{x \in \mathbb{Z} : \exists y \in \mathbb{Z}, x \sim y\} = \mathbb{Z}$.

(e) (4 points) **TRUE** or **FALSE** (circle one). If $\sim$ is an equivalence relation on a set A then the set of equivalence classes $A/\sim$ is a partition of the set A.

SOLUTION: TRUE. We explained this in class.

5
20 points