Final Exam

Intro to Discrete Math

MATH 2001

Spring 2022

Tuesday May 3, 2022

NAME: ______________________________

PRACTICE EXAM

SOLUTIONS

Question:	**1**	**2**	**3**	**4**	**5**	Total
Points:	20	20	20	20	20	100
Score:						

- The exam is closed book. You **may not use any resources** whatsoever, other than paper, pencil, and pen, to complete this exam.
- You **may not discuss the exam** with anyone except me, in any way, under any circumstances.
- You **must explain your answers**, and you will be **graded on the clarity of your solutions**.
- You must upload your exam as a single **.pdf** to **Canvas**, with the questions in the correct order, etc.
- You have 60 minutes to complete the exam.

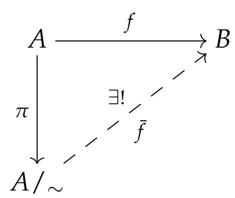
1. (20 points) • Let A be a set, and let $\sim$ be an equivalence relation on A.

Recall that for each $a \in A$, we denote by $[a]$ the equivalence class of A, and by $A/_\sim$ the set of equivalence classes of A. There is a map of sets $\pi : A \to A/_\sim$ defined by $a \mapsto [a]$.

Show that $\pi : A \to A/_\sim$ satisfies the following property:

> *If $f : A \to B$ is a map of sets such that for all $a_1, a_2 \in A$ with $a_1 \sim a_2$ we have $f(a_1) = f(a_2)$, then there exists a unique map of sets $\bar{f} : A/_\sim \to B$ such that $\bar{f} \circ \pi = f$.*

The following diagram may be helpful in thinking about this:



[*Hint*: show that the rule $\bar{f}([a]) = f(a)$ for all $[a] \in A/_\sim$ defines a map $\bar{f}$ with the desired properties.]

SOLUTION:

Solution. Let us first show that $\bar{f} : A/_\sim \to B$ defined by $\bar{f}([a]) = f(a)$ is a map of sets. To this end, I claim first that there is a set

$$\Gamma_{\bar{f}} = \{([a], f(a)) : [a] \in A/_\sim\} \subseteq (A/_\sim) \times B.$$

For this to make sense, I must show that the element $([a], f(a))$ depends only on $[a]$, and not on the choice of a. In other words, if $[a] = [a']$, then I must show that $([a], f(a)) = ([a'], f(a'))$. This is true since if $[a] = [a']$, then $a \sim a'$, and so, by assumption $f(a) = f(a')$, so that $([a], f(a)) = ([a'], f(a'))$. Thus $\Gamma_{\bar{f}}$ defines a subset of $(A/_\sim) \times B$.

Let us now show that $\Gamma_{\bar{f}}$ defines a map of sets. First, if we have $[a] \in A/_\sim$, and $b, b' \in B$, and $([a], b), ([a], b') \in \Gamma_{\bar{f}}$, then we have $b = f(a) = b'$. Second, if $[a] \in A/_\sim$, then $([a], f(a)) \in \Gamma_{\bar{f}}$. Thus $\Gamma_{\bar{f}}$ defines a map of sets $\bar{f} : A/_\sim \to B$. By definition, $\bar{f}([a]) = f(a)$.

Next let us show that $f = \bar{f} \circ \pi$. To this end, given $a \in A$, we have that $(\bar{f} \circ \pi)(a) = \bar{f}(\pi(a)) = \bar{f}([a]) = f(a)$.

Finally, let us show that $\bar{f}$ is unique. In other words, given any map of sets $g : A/_\sim \to B$ such that $g \circ \pi = f$, we must show that $g = \bar{f}$. Given $[a] \in A/_\sim$, we have $g([a]) = g(\pi(a)) = f(a) = \bar{f}(\pi(a)) = \bar{f}([a])$, and we are done. □

1
20 points

2. (20 points) • **TRUE** or **FALSE**:

Let S be a set, and let $\mathcal{A} \subseteq \mathscr{P}(S)$ be a set of subsets of S. Define a relation on $\mathcal{A}$ by the rule that for $A, B \in \mathcal{A}$, we have $A \leq B$ if $|A| \leq |B|$.

The relation $\leq$ gives $\mathcal{A}$ the structure of a POSET; i.e., $(\mathcal{A}, \leq)$ is a POSET.

If the statement in italics is true, give a proof. If the statement is false, provide a counter example, and prove that it is a counter example.

Your solution must start with the sentence, "*This statement is TRUE*," or the sentence, "*This statement is FALSE*."

SOLUTION:

Solution. This statement is FALSE.

While the relation $\leq$ is reflexive and transitive (see below), it need not be anti-symmetric; i.e., if $A \leq B$ and $B \leq A$, then we need not have $A = B$. Indeed, consider the example where $S = \{1,2\}$, $\mathcal{A} = \{\{1\}, \{2\}\}$, $A = \{1\}$, and $B = \{2\}$. Then $A \leq B$, and $B \leq A$, since $|A| = 1 = |B|$ so that $|A| \leq |B|$ and $|B| \leq |A|$; however, $A = \{1\} \neq \{2\} = B$. □

Although we do not need it for this problem, here is a proof of the assertion that $\leq$ is reflexive and transitive. First, the identity map $\mathrm{Id}_A : A \to A$ is an injection so that $|A| \leq |A|$, implying that $A \leq A$. Similarly, if $A \leq B$ and $B \leq C$, then we have $|A| \leq |B|$ and $|B| \leq |C|$, so that there are injections $f : A \to B$ and $g : B \to C$. The composition $g \circ f : A \to C$ is an injection, and therefore, $|A| \leq |C|$, so that $A \leq C$.

2
20 points

3. (20 points) • **TRUE** or **FALSE**:

> *For sets A and B, if there is an injection $f : A \to B$ and a surjection $g : A \to B$, then there is a bijection $h : A \to B$.*

If true, give a proof. If false, provide a counter example, and prove that it is a counter example.

Your solution must start with the sentence, "*This statement is TRUE*," or the sentence, "*This statement is FALSE*."

SOLUTION:

Solution. This statement is TRUE.

If there is an injection $f : A \to B$, then we have $|A| \leq |B|$. On the other hand, if we have a surjection $g : A \to B$, then, by the Axiom of Choice, there exists a section $s : B \to A$ (i.e., $g \circ s = \mathrm{Id}_B$). As we have seen, any section of a surjective map is injective (the composition $g \circ s$ is injective, so s is injective), and so we have $|B| \leq |A|$. Therefore, $|A| = |B|$ by the Cantor–Bernstein–Schroeder theorem; i.e., there exists a bijection $h : A \to B$. □

3
20 points

4. • A committee of 50 senators is chosen at random (from the full senate consisting of 100 senators). I want to be able to tell people what the probability is that any given collection of senators will be included in the committee.

(a) (10 points) **What probability space should I use to start to answer these types of questions?**

SOLUTION:

Solution. Let us number the senators from 1 to 100, and fix S to be the set of subsets of $\{1, 2, 3, \ldots, 100\}$ with order 50; i.e.,

$$S = \{A \subseteq \{1, 2, 3, \ldots, 100\} : |A| = 50\}.$$

Set $\mathcal{B} = \mathscr{P}(S)$, which we have seen is a Boolean algebra. Then, since $|S| = \binom{100}{50}$, we define $P : \mathcal{B} \to \mathbb{R}_{\geq 0}$ by the rule that for any $B \in \mathcal{B}$,

$$P(B) = \frac{1}{\binom{100}{50}} |B|.$$

On a homework exercise, we showed that this defines a probability measure on $(S, \mathcal{B})$, so that $(S, \mathcal{B}, P)$ is a probability space. This is the probability space I want to use. □

(b) (5 points) **If I give you a set of senators $C \subseteq \{1, 2, 3, \ldots, 100\}$, explain how you can use your probability space to determine the probability that the senators in C are included in the committee.** (For instance, you should be able to tell me the probability that both senators from Colorado are included in the committee.)

SOLUTION:

Solution. If you give me a set of senators $C \subseteq \{1, 2, 3, \ldots, 100\}$, and you ask me what is the probability that the senators in C are included in the committee, the answer is

$$P(\{A \in S : C \subseteq A\}) = \frac{1}{\binom{100}{50}} |\{A \in S : C \subseteq A\}|$$

and if $|C| = k \leq 50$, then $|\{A \in S : C \subseteq A\}| = \binom{100-k}{50-k}$, since once I choose the k senators that must be on the committee, then I must choose $50 - k$ more from the remaining $100 - k$. Therefore,

$$P(\{A \in S : C \subseteq A\}) = \frac{\binom{100-k}{50-k}}{\binom{100}{50}}.$$

□

(c) (5 points) **What is the probability that both senators from Colorado are included in the committee given that at least one is?**

SOLUTION:

Solution. Let A be the event that both senators from Colorado are chosen, and let B be the event that at least one of the two senators is chosen. We want to find $P(A|B) = \frac{P(A\cap B)}{P(B)}$. Since $A \subseteq B$, we have that $A \cap B = A$. Therefore we have $P(A|B) = \frac{P(A)}{P(B)}$. The event B has complement B^C, i.e., the event that neither senators are chosen. We have $P(B^C) = \frac{\binom{98}{50}}{\binom{100}{50}}$, since B^C can equivalently be viewed as the event that I have chosen a committee of 50 senators from the 98 other senators. Therefore, we have

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{P(A)}{1 - P(B^C)} = \frac{\frac{\binom{98}{48}}{\binom{100}{50}}}{1 - \frac{\binom{98}{50}}{\binom{100}{50}}} = \frac{\binom{98}{48}}{\binom{100}{50} - \binom{98}{50}}$$

□

4
20 points

5. • **TRUE** or **FALSE**. For this problem, and this problem only, **you do not need to justify your answer**.

(a) (4 points) **TRUE** or **FALSE** (circle one). The LaTeX code:

```
There exists $\bar f:A/_\sim \to B$ such that $\bar f([a])=f(a)$.
```

produces the following:

There exists $\bar f: A/_\sim \to B$ such that $\bar f([a]) = f(a)$.

SOLUTION: TRUE.

(b) (4 points) **TRUE** or **FALSE** (circle one). If $f: A \to B$ is a map of sets and $C \subseteq B$, then we have that $f(f^{-1}(C)) = C$.

SOLUTION: FALSE. Take $A = \{1\}$, $B = C = \{1,2\}$, and $f: A \to B$ given by $f(1) = 1$; then $f(f^{-1}(C) = \{1\} \neq C$.

(c) (4 points) **TRUE** or **FALSE** (circle one). Given a POSET $(P, \leq)$, a sub-POSET $(P', \leq')$ is called a chain if it has a maximal element.

SOLUTION: FALSE. A sub-POSET is called a chain if it is totally ordered.

(d) (4 points) **TRUE** or **FALSE** (circle one). The set $\mathrm{Map}(\mathbb{Q}, \mathbb{Q})$ of maps from $\mathbb{Q}$ to $\mathbb{Q}$ is countable.

SOLUTION: FALSE. We have seen that $\mathrm{Map}(\mathbb{N}, \{0,1\})$ is not countable, and there is an injection $\mathrm{Map}(\mathbb{N}, \{0,1\}) \hookrightarrow \mathrm{Map}(\mathbb{Q}, \mathbb{Q})$ given by sending $f: \mathbb{N} \to \{0,1\}$ to the map $g: \mathbb{Q} \to \mathbb{Q}$ given by $g(x) = x$ if $x \in \mathbb{N}$, and $g(x) = 0$ if $x \notin \mathbb{N}$.

(e) (4 points) **TRUE** or **FALSE** (circle one). If $(S, \mathcal{B}, P)$ is a probability space, and $A, B, C \in \mathcal{B}$ are events, then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

SOLUTION: TRUE. We use that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. We have $P(A \cup B \cup C) = P(A \cup B) + P(C) - P((A \cup B) \cap C) = P(A) + P(B) - P(A \cap B) + P(C) - P((A \cap C) \cup (B \cap C)) = P(A) + P(B) - P(A \cap B) + P(C) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$.

5
20 points