4.2. Partially ordered sets. It is often convenient to order things in a collection. This leads to the notion of a partially ordered set.

Definition 1.4.37. *A **POSET (partially ordered set)** consists of a set S and a subset $R \subseteq S \times S$ (a relation) such that for all $s_1, s_2, s_3 \in S$, the following hold:*

(1) *(Reflexive) $(s_1, s_1) \in R$;*
   (i.e., $s_1 \leq s_1$).
(2) *(Antisymmetric) If $(s_1, s_2) \in R$ and $(s_2, s_1) \in R$, then $s_1 = s_2$;*
   (i.e., $s_1 \leq s_2$ and $s_2 \leq s_1$ implies $s_1 = s_2$).
(3) *(Transitive) If $(s_1, s_2) \in R$ and $(s_2, s_3) \in R$, then $(s_1, s_3) \in R$;*
   (i.e., $s_1 \leq s_2$ and $s_2 \leq s_3$ implies $s_1 \leq s_3$).

REMARK 1.4.38. As indicated above, we will often write $s_1 \leq s_2$ if $(s_1, s_2) \in R$, and $(S, \leq)$ for (S, R).

REMARK 1.4.39. This definition is similar to that of an equivalence relation. The difference is in (2), where here we require that if $(s_1, s_2) \in R$ and $s_1 \neq s_2$, then $(s_2, s_1) \notin R$ (whereas for an equivalence relation we would require the opposite, that (s_2, s_1) was in the relation).

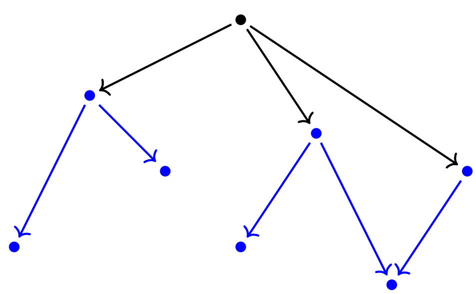

FIGURE 15. Diagram of a POSET

There are a number of notions related to POSETs that we will want to utilize.

- A **totally ordered set** is a POSET such that for every $s_1, s_2 \in S$, either $s_1 \leq s_2$ ($(s_1, s_2) \in R$) or $s_2 \leq s_1$ ($(s_2, s_1) \in R$).

  EXAMPLE 1.4.40. The real numbers with the usual notion of inequality is a totally ordered set.

- A **subPOSET** (S', R') of a POSET (S, R) is a POSET such that $S' \subseteq S$ and $R' = R \cap (S' \times S')$; in other words, for all $s'_1, s'_2 \in S'$, $s'_1 \leq' s'_2$ if and only if $s'_1 \leq s'_2$.

  EXAMPLE 1.4.41. The rational numbers inside of the real numbers form a subPOSET with the usual notion of inequality.

- An **upper bound** for a subPOSET (S', R') in (S, R) is an element $u \in S$ such that $s' \leq u$ ($(s', u) \in R$) for all $s' \in S'$.

  EXAMPLE 1.4.42. The negative real numbers form a subPOSET of the real numbers; this subPOSET has 1 as an upper bound. In fact any nonnegative real number will be an upper bound.

- A **chain** in a POSET (S, R) is a totally ordered subPOSET.
- A **maximal element** of a POSET (S, R) is an element $m \in S$ such that for all $s \in S$ we have $m \leq s$ ($(m, s) \in R$) implies $s = m$. (Note, this is not necessarily an upper bound for (S, R) in (S, R); i.e. there can be many maximal elements.)

Lemma 1.4.43 (Zorn's Lemma)**.** *Let $(S, \leq)$ be a POSET. If every chain in $(S, \leq)$ has an upper bound in $(S, \leq)$, then $(S, \leq)$ has a maximal element.*

REMARK 1.4.44. This is equivalent to the Axiom of Choice: Every surjective map of sets $X \to B$ admits a section. For a proof of the equivalence, see e.g. [**Mun00**].