Exercise 14.1.10

Introduction to Discrete Mathematics
MATH 2001

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ABSTRACT. This is Exercise 14.1.10 from Hammack [Ham13, §14.1]:

Exercise 14.1.10. Show that the two sets

$$\{0,1\} \times \mathbb{N} \text{ and } \mathbb{Z}$$

have equal cardinality by describing a bijection from one to the other. Describe your bijection with a formula (not as a table).

Solution. I claim that the map

$$f : \{0,1\} \times \mathbb{N} \longrightarrow \mathbb{Z}$$

$$f(a,b) = (-1)^a(b-a)$$

is a bijection, with inverse

$$g : \mathbb{Z} \longrightarrow \{0,1\} \times \mathbb{N}$$

$$g(z) = \begin{cases} (0,z) & z > 0 \\ (1,-z+1) & z \leq 0 \end{cases}$$

We prove the claim as follows. For all $(0,b) \in \{0,1\} \times \mathbb{N}$, we have $g(f(0,b)) = g(b) = (0,b)$, and for all $(1,b) \in \{0,1\} \times \mathbb{N}$ we have $g(f(1,b)) = g(-(b-1)) = (1,(b-1)+1) = (1,b)$. Thus $g \circ f = \mathrm{Id}_{\{0,1\}\times\mathbb{N}}$. Similarly, for all $z \in \mathbb{Z}$ with $z > 0$, we have $f(g(z)) = f(0,z) = z$, and for all $z \in \mathbb{Z}$ with $z \leq 0$, we have $f(g(z)) = f(1,-z+1) = (-1)(-z+1-1) = z$. Thus $f \circ g = \mathrm{Id}_{\mathbb{Z}}$. □

Remark 0.1. More intuitively (but less precisely), f is the map that identifies $\{0\} \times \mathbb{N} \subseteq \{0,1\} \times \mathbb{N}$ with the positive integers, and identifies $\{1\} \times \mathbb{N} \subseteq \{0,1\} \times \mathbb{N}$ with the negative integers shifted up by 1 (so that we do not miss zero).

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References

[Ham13] Richard Hammack, *Book of proof*, Creative Commons, 2013.

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