Exercise 30.4

Abstract Algebra 1
MATH 3140

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ABSTRACT. This is Exercise 30.4 from Fraleigh [Fra03, §30]:

Exercise 30.4. Give a basis for the vector space $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$.

Solution. A basis for $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$ is given by $\{1, \sqrt{2}\}$. This follows from [Fra03, Theorem 30.23]. Indeed, I claim that $\operatorname{irr}(\sqrt{2}, \mathbb{Q}) = x^2 - 2$. Certainly $\sqrt{2}$ is a root of $x^2 - 2$, and one can see that $x^2 - 2$ is irreducible, since if it factored, it would factor into linear terms, and then there would be a rational square root of 2, which we know is not the case. Thus $\deg(\sqrt{2}, \mathbb{Q}) = 2$, and [Fra03, Theorem 30.23] implies that $\{1, \sqrt{2}\}$ is a basis. $\square$

Remark 0.1. Here is a direct argument for the exercise, which recapitulates the proof of [Fra03, Theorem 30.23] in the situation of this exercise. From [Fra03, Case I, p.270], we know that the evaluation homomorphism $\phi_{\sqrt{2}}$ gives an isomorphism of $\mathbb{Q}[x]/\langle \operatorname{irr}(\sqrt{2}, \mathbb{Q})\rangle$ with $\mathbb{Q}(\sqrt{2})$. Therefore, since $\operatorname{irr}(\sqrt{2}, \mathbb{Q}) = x^2 - 2$ (as explained above), the evaluation homomorphism $\phi_{\sqrt{2}}$ gives an isomorphism

$$\frac{\mathbb{Q}[x]}{\langle x^2 - 2\rangle} \xrightarrow{\sim} \mathbb{Q}(\sqrt{2}). \tag{0.1}$$

Now I claim the classes of 1 and x form a basis for $\mathbb{Q}[x]/\langle x^2 - 2\rangle$. First, the classes of 1 and x span $\mathbb{Q}[x]/\langle x^2 - 2\rangle$, since the class of any polynomial $a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ is equal to the class of $a_0 + a_1x + 2a_2 + 2a_3x + 2^2a_4 + \cdots = (a_0 + 2a_2 + 2^2a_4 + \cdots) \cdot 1 + (a_1 + 2a_3 + 2^2a_5 + \cdots) \cdot x$.

Second, the classes of 1 and x are linearly independent in $\mathbb{Q}[x]/\langle x^2 - 2\rangle$, since if there were a linear relation $a \cdot 1 + b \cdot x = 0$ in $\mathbb{Q}[x]/\langle x^2 - 2\rangle$, then $a + bx$ would be in the kernel of the evaluation map $\phi_{\sqrt{2}} : \mathbb{Q}[x] \to \mathbb{Q}(\sqrt{2})$, contradicting the fact that $\operatorname{irr}(\sqrt{2}, \mathbb{Q})$ is the monic polynomial of minimal degree in the kernel of the evaluation map.

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Since the classes of 1 and x form a basis for $\mathbb{Q}[x]/\langle x^2 - 2\rangle$, their images under the isomorphism (0.1) form a basis for $\mathbb{Q}(\sqrt{2})$. Their images are 1 and $\sqrt{2}$ respectively, and so 1 and $\sqrt{2}$ form a basis for $\mathbb{Q}(\sqrt{2})$.

REFERENCES

[Fra03] John Fraleigh, *A First Course in Abstract Algebra*, Seventh edition, Addison Wesley, Pearson, 2003.

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