Exercise 29.33

Abstract Algebra 1
MATH 3140

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ABSTRACT. This is Exercise 29.33 from Fraleigh [Fra03, §29]:

Exercise 29.33. Let E be an extension of a field F and let $\alpha \in E$ be transcendental over F. Show that every element of $F(\alpha)$ that is not in F is also transcendental over F.

Solution. Denote by $F(x)$ the field of fractions ("quotients") of the polynomial ring $F[x]$. From [Fra03, Case II, p.270], since α is transcendental over F, the evaluation homomorphism $\phi_a : F[x] \to E$ induces an isomorphism

$$\phi_a : F(x) \xrightarrow{\sim} F(\alpha).$$

In other words, it suffices to show that if $\dfrac{p(x)}{q(x)} \in F(x)$ (for some $p(x), q(x) \in F[x]$ with $q(x) \neq 0$) is algebraic over F, then $\dfrac{p(x)}{q(x)}$ is contained in F (i.e., is a ratio of constant polynomials).

Note that using [Fra03, Theorem 23.20], we may factor $p(x)$ and $q(x)$ into irreducibles, and therefore, cancelling irreducible factors, we may and will assume that $p(x)$ and $q(x)$ have no common irreducible factors.

Now let us assume that $\dfrac{p(x)}{q(x)}$ is algebraic over F; i.e., it satisfies a monic polynomial

$$T^n + a_{n-1}T^{n-1} + \cdots + a_1 T + a_0 \in F[T]. \tag{0.1}$$

In other words, assume

$$\left(\frac{p(x)}{q(x)}\right)^n + a_{n-1}\left(\frac{p(x)}{q(x)}\right)^{n-1} + \cdots + a_1\left(\frac{p(x)}{q(x)}\right) + a_0 = 0.$$

Multiplying by $q(x)^n$, we have

$$p(x)^n + a_{n-1}p(x)^{n-1}q(x) + \cdots + a_1 p(x)q(x)^{n-1} + a_0 q(x)^n = 0,$$

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which we may rewrite as

$$p(x)^n = -(a_{n-1}p(x)^{n-1}q(x) + \cdots + a_1 p(x)q(x)^{n-1} + a_0 q(x)^n).$$

Since the right hand side is divisible by $q(x)$, the left hand side must also be divisible by $q(x)$. But we have assumed that $p(x)$ and $q(x)$ have no common irreducible factors, so it must be that $q(x)$ is a nonzero constant (i.e., in F^*). In other words, $g(x) := p(x)/q(x) \in F[x]$ is a polynomial, which satisfies the monic polynomial (0.1), above.

We want to show $g(x) \in F$. If $g(x) = 0$, then clearly $g(x) \in F$, so let us assume that $g(x) \neq 0$. Then we can write

$$g(x) = b_e x^e + \cdots + b_1 x + b_0, \quad b_e \neq 0,$$

for some nonnegative integer e. We want to show that $e = 0$. For the sake of contradiction, assume that $e \neq 0$. Then substituting into (0.1) we have

$$\begin{aligned} 0 &= (b_e x^e + \cdots + b_1 x + b_0)^n + a_{n-1}(b_e x^e + \cdots + b_1 x + b_0)^{n-1} + \cdots + a_0 \\ &= b_e^n x^{ne} + \text{ lower order terms in } x. \end{aligned}$$

This is not possible, since $b_e \neq 0$, and therefore our assumption that $e \neq 0$ was false. Thus $e = 0$, and $g(x)$ is a constant. □

REFERENCES

[Fra03] John Fraleigh, *A First Course in Abstract Algebra*, Seventh edition, Addison Wesley, Pearson, 2003.

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