Midterm 2

Linear Algebra: Matrix Methods

MATH 2130

Fall 2022

Friday October 28, 2022

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NAME: ______________________________

PRACTICE EXAM

SOLUTIONS

Question:	**1**	**2**	**3**	**4**	**5**	Total
Points:	20	20	20	20	20	100
Score:						

- The exam is closed book. You **may not use any resources** whatsoever, other than paper, pencil, and pen, to complete this exam.
- You **may not discuss the exam** with anyone except me, in any way, under any circumstances.
- You **must explain your answers**, and you will be **graded on the clarity of your solutions**.
- You must upload your exam as a single **.pdf** to **Canvas**, with the questions in the correct order, etc.
- You have 45 minutes to complete the exam.

1. • *Compute the determinant of each of the following matrices:*

(a) (10 points) $A = \begin{pmatrix} 4 & -1 & 1 \\ -1 & -2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

SOLUTION:

Solution. We have $\boxed{\det A = -1}$ The fastest way to see this may be to expand off of the third column (or even to interchange two columns, twice); however, to use the standard method, we have

$$\det A = (4)[(-2)(0) - (0)(1)] - (-1)[(-1)(0) - (0)(0)] + (1)[(-1)(1) - (-2)(0)] = -1.$$

□

(b) (10 points) $B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \pi \\ 1 & 0 & e & -4 & 8 & 3^{-5} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 & 2 & 10^4 \\ 0 & 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & -1 & 2 & 0 \end{pmatrix}$

SOLUTION:

Solution. We have $\boxed{\det B = -2}$ We use row operations:

$$\begin{vmatrix} 0 & 1 & 0 & 0 & 0 & \pi \\ 1 & 0 & e & -4 & 8 & 3^{-5} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 & 2 & 10^4 \\ 0 & 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & -1 & 2 & 0 \end{vmatrix} = (-1)^2 \begin{vmatrix} 1 & 0 & e & -4 & 8 & 3^{-5} \\ 0 & 1 & 0 & 0 & 0 & \pi \\ 0 & 5 & 1 & 0 & 2 & 10^4 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & -1 & 2 & 0 \end{vmatrix} = (-1)^2 \begin{vmatrix} 1 & 0 & e & -4 & 8 & 3^{-5} \\ 0 & 1 & 0 & 0 & 0 & \pi \\ 0 & 0 & 1 & 0 & 2 & 10^4 - 5\pi \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & -1 & 2 & 0 \end{vmatrix}$$

$$= (-1)^2 \begin{vmatrix} 1 & 0 & e & -4 & 8 & 3^{-5} \\ 0 & 1 & 0 & 0 & 0 & \pi \\ 0 & 0 & 1 & 0 & 2 & 10^4 - 5\pi \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{vmatrix} = (-1)^3 \begin{vmatrix} 1 & 0 & e & -4 & 8 & 3^{-5} \\ 0 & 1 & 0 & 0 & 0 & \pi \\ 0 & 0 & 1 & 0 & 2 & 10^4 - 5\pi \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = -2$$

□

1
20 points

2. (20 points) • Let V_1 and V_2 be real vector spaces. On the product

$$V_1 \times V_2 = \{(\mathbf{v}_1, \mathbf{v}_2) : \mathbf{v}_1 \in V_1,\ \mathbf{v}_2 \in V_2\},$$

define addition and scaling rules by

$$(\mathbf{v}_1, \mathbf{v}_2) + (\mathbf{w}_1, \mathbf{w}_2) = (\mathbf{v}_1 + \mathbf{w}_1, \mathbf{v}_2 + \mathbf{w}_2)$$

$$\lambda \cdot (\mathbf{v}_1, \mathbf{v}_2) = (\lambda \cdot \mathbf{v}_1, \lambda \cdot \mathbf{v}_2).$$

Show that these addition and scaling rules make $V_1 \times V_2$ into a real vector space.

SOLUTION:

Solution. For brevity of notation, I will write $V = V_1 \times V_2$.

1. (Group laws)

   (a) (Additive identity) I claim there exists an element $\mathscr{O} \in V$ such that for all $\mathbf{v} \in V$, $\mathbf{v} + \mathscr{O} = \mathbf{v}$.

   Indeed, set $\mathscr{O} = (\mathscr{O}_1, \mathscr{O}_2)$, where $\mathscr{O}_1 \in V_1$ is the additive identity for V_1 and $\mathscr{O}_2 \in V_2$ is the additive identity for V_2. Then for any $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2) \in V = V_2 \times V_2$, we have

$$\begin{aligned}
\mathbf{v} + \mathscr{O} &= (\mathbf{v}_1, \mathbf{v}_2) + (\mathscr{O}_1, \mathscr{O}_2) \\
&= (\mathbf{v}_1 + \mathscr{O}_1, \mathbf{v}_2 + \mathscr{O}_2) && \text{Def. of } + \text{ in } V \\
&= (\mathbf{v}_1, \mathbf{v}_2) && \text{(1)(a) for } V_1 \text{ and } V_2 \\
&= \mathbf{v}.
\end{aligned}$$

   (b) (Additive inverse) I claim that for each $\mathbf{v} \in V$ there exists an element $-\mathbf{v} \in V$ such that $\mathbf{v} + (-\mathbf{v}) = \mathscr{O}$.

   Indeed, given $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2) \in V = V_1 \times V_2$, set $-\mathbf{v} = (-\mathbf{v}_1, -\mathbf{v}_2)$, where $-\mathbf{v}_1 \in V_1$ is the additive inverse of $\mathbf{v}_1$, and $-\mathbf{v}_2 \in V_2$ is the additive inverse of $\mathbf{v}_2$. Then

$$\begin{aligned}
\mathbf{v} + (-\mathbf{v}) &= (\mathbf{v}_1, \mathbf{v}_2) + (-\mathbf{v}_1, -\mathbf{v}_2) \\
&= (\mathbf{v}_1 + (-\mathbf{v}_1), \mathbf{v}_2 + (-\mathbf{v}_2)) && \text{Def. of } + \text{ in } V \\
&= (\mathscr{O}_1, \mathscr{O}_2) && \text{(1)(b) for } V_1 \text{ and } V_2 \\
&= \mathscr{O}.
\end{aligned}$$

(c) (Associativity of addition) I claim that for all $u, \mathbf{v}, w \in V$,

$$(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w}).$$

Indeed, given $\mathbf{u}=(\mathbf{u}_1,\mathbf{u}_2), \mathbf{v}=(\mathbf{v}_1,\mathbf{v}_2), \mathbf{w}=(\mathbf{w}_1,\mathbf{w}_2) \in V = V_1 \times V_2$, we have

$$\begin{aligned}
(\mathbf{u}+\mathbf{v})+\mathbf{w} &= ((\mathbf{u}_1,\mathbf{u}_2)+(\mathbf{v}_1,\mathbf{v}_2))+(\mathbf{w}_1,\mathbf{w}_2) \\
&= (\mathbf{u}_1+\mathbf{v}_1,\mathbf{u}_2+\mathbf{v}_2)+(\mathbf{w}_1,\mathbf{w}_2) && \text{Def. of } + \text{ in } V \\
&= ((\mathbf{u}_1+\mathbf{v}_1)+\mathbf{w}_1,(\mathbf{u}_2+\mathbf{v}_2)+\mathbf{w}_2) && \text{Def. of } + \text{ in } V \\
&= (\mathbf{u}_1+(\mathbf{v}_1+\mathbf{w}_1),\mathbf{u}_2+(\mathbf{v}_2+\mathbf{w}_2)) && \text{(1)(c) for } V_1 \text{ and } V_2 \\
&= (\mathbf{u}_1,\mathbf{u}_2)+(\mathbf{v}_1+\mathbf{w}_1,\mathbf{v}_2+\mathbf{w}_2) && \text{Def. of } + \text{ in } V \\
&= (\mathbf{u}_1,\mathbf{u}_2)+((\mathbf{v}_1,\mathbf{v}_2)+(\mathbf{w}_1,\mathbf{w}_2)) && \text{Def. of } + \text{ in } V \\
&= \mathbf{u}+(\mathbf{v}+\mathbf{w}).
\end{aligned}$$

2. (Abelian property)

(a) (Commutativity of addition) For all $\mathbf{u},\mathbf{v} \in V$,

$$\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}.$$

Indeed, given $\mathbf{u}=(\mathbf{u}_1,\mathbf{u}_2),\ \mathbf{v}=(\mathbf{v}_1,\mathbf{v}_2) \in V = V_1 \times V_2$, we have

$$\begin{aligned}
u+v &= (\mathbf{u}_1,\mathbf{u}_2)+(\mathbf{v}_1,\mathbf{v}_2) \\
&= (\mathbf{u}_1+\mathbf{v}_1,\mathbf{u}_2+\mathbf{v}_2) && \text{Def. of } + \text{ in } V \\
&= (\mathbf{v}_1+\mathbf{u}_1,\mathbf{v}_2+\mathbf{u}_2) && \text{(2)(a) for } V_1 \text{ and } V_2 \\
&= (\mathbf{v}_1,\mathbf{v}_2)+(\mathbf{u}_1,\mathbf{u}_2) && \text{Def. of } + \text{ in } V \\
&= \mathbf{v}+\mathbf{u}.
\end{aligned}$$

3. (Module conditions)

(a) I claim that for all $\lambda \in K$ and all $\mathbf{u},\mathbf{v} \in V$,

$$\lambda \cdot (\mathbf{u}+\mathbf{v}) = (\lambda \cdot \mathbf{u}) + (\lambda \cdot \mathbf{v}).$$

Indeed, given $\lambda \in K$ and $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2),\ \mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2) \in V = V_1 \times V_2$, we have

$$\begin{aligned}
\lambda \cdot (\mathbf{u} + \mathbf{v}) &= \lambda \cdot ((\mathbf{u}_1, \mathbf{u}_2) + (\mathbf{v}_1, \mathbf{v}_2)) \\
&= \lambda \cdot (\mathbf{u}_1 + \mathbf{v}_1, \mathbf{u}_2 + \mathbf{v}_2) && \text{Def. of } + \text{ in } V \\
&= (\lambda \cdot (\mathbf{u}_1 + \mathbf{v}_1), \lambda \cdot (\mathbf{u}_2 + \mathbf{v}_2)) && \text{Def. of } \cdot \text{ in } V \\
&= (\lambda \cdot \mathbf{u}_1 + \lambda \cdot_1 \mathbf{v}_1, \lambda \cdot \mathbf{u}_2 + \lambda \cdot \mathbf{v}_2) && \text{(3)(a) for } V_1 \text{ and } V_2 \\
&= (\lambda \cdot \mathbf{u}_1, \lambda \cdot \mathbf{u}_2) + (\lambda \cdot_1 \mathbf{v}_1, \lambda \cdot \mathbf{v}_2) && \text{Def. of } + \text{ in } V \\
&= \lambda \cdot (\mathbf{u}_1, \mathbf{u}_1) + \lambda \cdot (\mathbf{v}_1, \mathbf{v}_2) && \text{Def. of } \cdot \text{ in } V \\
&= (\lambda \cdot \mathbf{u}) + (\lambda \cdot \mathbf{v})
\end{aligned}$$

(b) I claim that for all $\lambda, \mu \in K$, and all $\mathbf{v} \in V$,

$$(\lambda + \mu) \cdot \mathbf{v} = (\lambda \cdot \mathbf{v}) + (\mu \cdot \mathbf{v}).$$

Indeed, given $\lambda, \mu \in K$ and $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2) \in V = V_1 \times V_2$, we have

$$\begin{aligned}
(\lambda + \mu) \cdot \mathbf{v} &= (\lambda + \mu) \cdot (\mathbf{v}_1, \mathbf{v}_2) \\
&= ((\lambda + \mu) \cdot \mathbf{v}_1, (\lambda + \mu) \cdot \mathbf{v}_2) && \text{Def. of } \cdot \text{ in } V \\
&= (\lambda \cdot \mathbf{v}_1 + \mu \cdot_1 \mathbf{v}_1, \lambda \cdot \mathbf{v}_2 + \mu \cdot \mathbf{v}_2) && \text{(3)(b) for } V_1 \text{ and } V_2 \\
&= (\lambda \cdot \mathbf{v}_1, \lambda \cdot \mathbf{v}_2) + (\mu \cdot \mathbf{v}_1, \mu \cdot_2 \mathbf{v}_2) && \text{Def. of } + \text{ in } V \\
&= \lambda \cdot (\mathbf{v}_1, \mathbf{v}_2) + \mu \cdot (\mathbf{v}_1, \mathbf{v}_2) && \text{Def. of } \cdot \text{ in } V \\
&= (\lambda \cdot \mathbf{v}) + (\mu \cdot \mathbf{v}).
\end{aligned}$$

(c) For all $\lambda, \mu \in K$, and all $\mathbf{v} \in V$,

$$(\lambda\mu) \cdot \mathbf{v} = \lambda \cdot (\mu \cdot \mathbf{v}).$$

Indeed, given $\lambda, \mu \in K$ and $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2) \in V = V_1 \times V_2$, we have

$$\begin{aligned}
(\lambda\mu) \cdot \mathbf{v} &= (\lambda\mu) \cdot (\mathbf{v}_1, \mathbf{v}_2) \\
&= ((\lambda\mu) \cdot \mathbf{v}_1, (\lambda\mu) \cdot \mathbf{v}_2) && \text{Def. of } \cdot \text{ in } V \\
&= (\lambda \cdot (\mu \cdot_1 \mathbf{v}_1), \lambda \cdot (\mu \cdot \mathbf{v}_2)) && \text{(3)(c) for } V_1 \text{ and } V_2 \\
&= \lambda \cdot (\mu \cdot \mathbf{v}_1, \mu \cdot \mathbf{v}_2) && \text{Def. of } \cdot \text{ in } V \\
&= \lambda \cdot (\mu \cdot (\mathbf{v}_1, \mathbf{v}_2)) && \text{Def. of } \cdot \text{ in } V \\
&= \lambda \cdot (\mu \cdot \mathbf{v}).
\end{aligned}$$

(d) I claim that for all $\mathbf{v} \in V$,

$$1 \cdot \mathbf{v} = \mathbf{v}.$$

Indeed, given $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2) \in V = V_1 \times V_2$, we have

$$\begin{aligned}
1 \cdot \mathbf{v} &= 1 \cdot (\mathbf{v}_1, \mathbf{v}_2) \\
&= (1 \cdot \mathbf{v}_1, 1 \cdot \mathbf{v}_2) && \text{Def. of } \cdot \text{ in } V \\
&= (\mathbf{v}_1, \mathbf{v}_2) && \text{(3)(d) for } V_1 \text{ and } V_2 \\
&= \mathbf{v}.
\end{aligned}$$

□

2
20 points

3. (20 points) • Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ be bases for a real vector space V, and suppose that

$$\begin{aligned} \mathbf{v}_1 &= 4\mathbf{w}_1 - \mathbf{w}_2 + \mathbf{w}_3 \\ \mathbf{v}_2 &= 3\mathbf{w}_1 + 2\mathbf{w}_2 - \mathbf{w}_3 \\ \mathbf{v}_3 &= 7\mathbf{w}_1 + 23\mathbf{w}_2 - 2\mathbf{w}_3 \end{aligned}$$

Find the change-of-coordinates matrix to go from the coordinates with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ *to the coordinates with respect to the basis* $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

SOLUTION:

Solution. The change-of-coordinates matrix to go from the coordinates with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to the coordinates with respect to the basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$can be read off from the equations above as the matrix

$$\begin{bmatrix} 4 & -1 & 1 \\ 3 & 2 & -1 \\ 7 & 23 & -2 \end{bmatrix}^T = \boxed{\begin{bmatrix} 4 & 3 & 7 \\ -1 & 2 & 23 \\ 1 & -1 & -2 \end{bmatrix}}.$$

□

3
20 points

4. • Consider the 2-dimensional discrete dynamical system

$$\mathbf{x}_{k+1} = A\mathbf{x}_k$$

where

$$A = \begin{pmatrix} 1.7 & 0.3 \\ 1.2 & 0.8 \end{pmatrix}$$

(a) (10 points) *Is the origin an attractor, repeller, or saddle point?*

SOLUTION:

Solution. The origin is a saddle point.

To see this, we compute that the characteristic polynomial is

$$p_A(t) = \det \begin{pmatrix} t-1.7 & -0.3 \\ -1.2 & t-0.8 \end{pmatrix} = (t^2 - 2.5t + 1.36) - (.36) = t^2 - 2.5t + 1$$

$$= (t-2)(t-\frac{1}{2})$$

Thus the eigenvalues are $\lambda = \frac{1}{2}, 2$. Since $0 < \frac{1}{2} < 1$ and $1 < 2$, we see that the origin is a saddle point. □

(b) (10 points) *Find the directions of greatest attraction or repulsion.*

SOLUTION:

Solution. We have that the line spanned by $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ is the direction of greatest attraction, and the line spanned by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the direction of greatest repulsion.

To deduce this, we find the eigenspaces. We start with the $\lambda = \frac{1}{2}$-eigenspace, $E_{1/2}$, which is the kernel of $\frac{1}{2}I - A$:

$$\frac{1}{2}I - A = \begin{pmatrix} -1.2 & -0.3 \\ -1.2 & -0.3 \end{pmatrix} \mapsto \begin{pmatrix} 12 & 3 \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1/4 \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1/4 \\ {\color{red}0} & {\color{blue}-1} \end{pmatrix}$$

Thus $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ is a basis for the $\frac{1}{2}$-eigenspace $E_{1/2}$.

We now compute the $\lambda = 2$-eigenspace, E_2, which is the kernel of $2I - A$:

$$2I - A = \begin{pmatrix} 0.3 & -0.3 \\ -1.2 & 1.2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -1 \\ {\color{red}0} & {\color{blue}-1} \end{pmatrix}$$

Thus $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a basis for the 2-eigenspace E_2.

In conclusion, the line spanned by $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ is the direction of greatest attraction, and the line spanned by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the direction of greatest repulsion. □

4
20 points

5. • Consider the following real matrix

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & 1 & 3 \end{pmatrix}$$

(a) (5 points) *Find the characteristic polynomial $p_A(t)$ of A.*

SOLUTION:

Solution. We have

$$\begin{aligned} p_A(t) &= \begin{vmatrix} t-2 & 1 & -1 \\ 0 & t-3 & 1 \\ -2 & -1 & t-3 \end{vmatrix} \\ &= (t-2)[(t-3)^2 - (1)(-1)] - (1)[0-(1)(-2)] + (-1)[0-(t-3)(-2)] \\ &= (t-2)[t^2-6t+10] - 2 + \underbrace{(t-3)(-2)}_{-2t+6} \\ &= (t^3 - 6t^t + 10t - 2t^2 + 12t - 20) - 2 + (6-2t) \\ &= t^3 - 8t^2 + 20t - 16. \end{aligned}$$

In other words, the solution is:

$$\boxed{p_A(t) = t^3 - 8t^2 + 20t - 16.}$$

□

As a quick partial check of the solution, observe that

$$\begin{aligned} \operatorname{tr}(A) &= 8 \\ \det A &= \begin{vmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 0 & 2 & 2 \end{vmatrix} = 2(6+2) = 16. \end{aligned}$$

confirming the computation of the coefficients of t^2 and t^0, since we know that

$$p_A(t) = t^3 - \mathrm{tr}(A)t^2 + \alpha t + (-1)^3 \det(A)$$

for some real number $\alpha \in \mathbb{R}$.

(b) (5 points) *Find the eigenvalues of A.*

SOLUTION:

Solution. One can easily check that

$$p_A(2) = 2^3 - 8 \cdot 2^2 + 20 \cdot 2 - 16 = 8 - 32 + 40 - 16 = 48 - 48 = 0.$$

Thus $(t-2)$ is a factor of $p_A(t)$, so that we have

$$p_A(t) = (t-2)(t^2 - 6t + 8) = (t-2)(t-2)(t-4).$$

Thus the eigenvalues are

$$\boxed{\lambda = 2, 4.}$$

□

(c) (5 points) *Find a basis for each eigenspace of A in $\mathbb{R}^3$.*

SOLUTION:

Solution. To find a basis for the $\lambda = 2$ eigenspace E_2, we compute

$$E_2 := \ker(2I - A) = \ker \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ -2 & -1 & -1 \end{pmatrix}$$

$$= \ker \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} = \ker \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \ker \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \ker \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

We add rows, and get the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}$$

Thus we have

$$E_2 = \left\{ \alpha \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} : \alpha \in \mathbb{R} \right\}$$

Now we compute a basis for the $\lambda = 4$ eigenspace E_4. We have

$$E_4 = \ker \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ -2 & -1 & 1 \end{pmatrix} = \ker \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \ker \begin{pmatrix} 2 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \ker \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

This gives us the matrix

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

Thus we have

$$E_4 = \left\{ \alpha \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} : \alpha \in \mathbb{R} \right\}$$

Thus the solution to the problem is:

The eigenspaces for A are E_2 and E_4, and we have that $\boxed{\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}}$ is a basis for E_2 and $\boxed{\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}}$ is a basis for E_4. □

(d) (5 points) *Is A diagonalizable? If so, find a matrix $S \in \mathrm{M}_{3\times 3}(\mathbb{R})$ so that $S^{-1}AS$ is diagonal. If not, explain.*

SOLUTION:

Solution. **No.** A is not diagonalizable since we showed in part (c) that there does not exists a basis of $\mathbb{R}^3$ consisting of eigenvectors for A. □

5
20 points