Final Exam

Linear Algebra: Matrix Methods

MATH 2130

Fall 2022

Sunday December 11, 2022

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NAME: ____

PRACTICE EXAM

SOLUTIONS

Question:	1	2	3	4	5	6	7	Total
Points:	20	20	20	20	20	20	20	140
Score:								

- The exam is closed book. You **may not use any resources** whatsoever, other than paper, pencil, and pen, to complete this exam.
- You **may not discuss the exam** with anyone except me, in any way, under any circumstances.
- You **must explain your answers**, and you will be **graded on the clarity of your solutions**.
- You must upload your exam as a single **.pdf** to **Canvas**, with the questions in the correct order, etc.
- You have 70 minutes to complete the exam.

1. (20 points) • Let $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{x}_4 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$.

Use the Gram–Schmidt process to find an orthonormal basis for the vector subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, and $\mathbf{x}_4$.

SOLUTION:

Solution. An orthonormal basis is given by

$$\boxed{\mathbf{u}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sqrt{15}}\begin{bmatrix} -1 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \quad \mathbf{u}_3 = \frac{1}{\sqrt{35}}\begin{bmatrix} 1 \\ 3 \\ -3 \\ -4 \end{bmatrix}}$$

We start by finding an orthogonal basis. We have

$$\mathbf{y}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{y}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 . \mathbf{y}_1}{\mathbf{y}_1 . \mathbf{y}_1}\mathbf{y}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 2/3 \\ 1 \\ -1/3 \end{bmatrix} \sim \begin{bmatrix} -1 \\ 2 \\ 3 \\ -1 \end{bmatrix}$$

For simplicity, we will take

$$\mathbf{y}_2 = \begin{bmatrix} -1 \\ 2 \\ 3 \\ -1 \end{bmatrix}$$

We have

$$\mathbf{y}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3.\mathbf{y}_1}{\mathbf{y}_1.\mathbf{y}_1}\mathbf{y}_1 - \frac{\mathbf{x}_3.\mathbf{y}_2}{\mathbf{y}_2.\mathbf{y}_2}\mathbf{y}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{15}\begin{bmatrix} -1 \\ 2 \\ 3 \\ -1 \end{bmatrix} =$$

$$= \frac{1}{15}\begin{bmatrix} 0 \\ 0 \\ 15 \\ 15 \end{bmatrix} + \frac{1}{15}\begin{bmatrix} -5 \\ -5 \\ 0 \\ -5 \end{bmatrix} + \frac{1}{15}\begin{bmatrix} 2 \\ -4 \\ -6 \\ 2 \end{bmatrix} = \frac{1}{15}\begin{bmatrix} -3 \\ -9 \\ 9 \\ 12 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 3 \\ -3 \\ -4 \end{bmatrix}$$

Again for simplicity we take

$$\mathbf{y}_3 = \begin{bmatrix} 1 \\ 3 \\ -3 \\ -4 \end{bmatrix}$$

Note that since $\mathbf{x}_4 = \mathbf{x}_1 + \mathbf{x}_2 - \mathbf{x}_3$, we see that $\mathbf{x}_4$ is in the span of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, so if we perform Gram–Schmidt to $\mathbf{x}_4$, we will get $\mathbf{y}_4 = 0$. I omit the computation here for brevity (but you should check!).

Therefore, an orthogonal basis for the span of $\mathbf{x}_1, \ldots, \mathbf{x}_4$ is given by

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} -1 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} 1 \\ 3 \\ -3 \\ -4 \end{bmatrix}$$

Consequently, an orthonormal basis is given by

$$\mathbf{u}_1 = \frac{\mathbf{y}_1}{\|\mathbf{y}_1\|} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \frac{\mathbf{y}_2}{\|\mathbf{y}_2\|} = \frac{1}{\sqrt{15}}\begin{bmatrix} -1 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \quad \mathbf{u}_3 = \frac{\mathbf{y}_3}{\|\mathbf{y}_3\|} = \frac{1}{\sqrt{35}}\begin{bmatrix} 1 \\ 3 \\ -3 \\ -4 \end{bmatrix}$$

$\square$

2. (20 points) • Let $\mathbb{P}_3$ be the real vector space of polynomials of degree at most 3 (my notation for this vector space has been $\mathbb{R}[t]_3$, but here I am using the textbook's notation). A basis of $\mathbb{P}_3$ is given by the polynomials $1, t, t^2, t^3$.

We have seen that there is an inner product on $\mathbb{P}_3$ given by evaluation at -2, -1, 1, and 2. In other words, given polynomials $p(t), q(t) \in \mathbb{P}_3$, we define the inner product by the rule

$$\begin{aligned}(p(t), q(t)) &:= (p(-2), p(-1), p(1), p(2)).(q(-2), q(-1), q(1), q(2)) \\ &= p(-2)q(-2) + p(-1)q(-1) + p(1)q(1) + p(2)q(2).\end{aligned}$$

Let $p_1(t) = t$, and $p_2(t) = t^2$.

Find the best approximation to $p(t) = t^3$ *by the polynomials in* $\operatorname{Span}\{p_1(t), p_2(t)\}$.

In other words, find the polynomial $q(t)$ in the span of $p_1(t)$ and $p_2(t)$, that is closest to the polynomial $p(t)$ with respect to the given inner product on $\mathbb{P}_3$.

SOLUTION:

Solution. The best approximation to $p(t) = t^3$ by the polynomials in $\operatorname{Span}\{p_1(t), p_2(t)\}$ is

$$\boxed{q(t) = \frac{17}{5}t.}$$

To show this, we need to compute the orthogonal projection of $p(t) = t^3$ onto $\operatorname{Span}\{p_1(t), p_2(t)\} = \operatorname{Span}\{t, t^2\}$.

First we will construct an orthonormal basis for $\operatorname{Span}\{t, t^2\}$ by performing Gram–Schmidt on the given basis $p_1(t) = t$, $p_2(t) = t^2$.

For this problem, it is convenient to make the following table

	-2	-1	1	2
$p_1(t) = t$	$(-2,$	$-1,$	$1,$	$2)$
$p_2(t) = t^2$	$(4,$	$1,$	$1,$	$4)$
$p(t) = t^3$	$(-8,$	$-1,$	$1,$	$8)$

Then the inner product in $\mathbb{P}_3$ is given by dotting the corresponding vectors above. In other words, we have

$$
\begin{aligned}
(p_1(t), p_1(t)) &= (-2, -1, 1, 2).(-2, -1, 1, 2) = 10 \\
(p_1(t), p_2(t)) &= (-2, -1, 1, 2).(4, 1, 1, 4) = 0 \\
(p_1(t), p(t)) &= 34 \\
(p_2(t), p_2(t)) &= 34 \\
(p_2(t), p(t)) &= 0
\end{aligned}
$$

Fortunately, we see that the basis $p_1(t)$, $p_2(t)$, is already orthogonal. Thus we can compute the projection $q(t)$ of $p(t)$ onto $\text{Span}\{p_1(t), p_2(t)\}$ as

$$
\begin{aligned}
q(t) &= \frac{(p(t), p_1(t))}{(p_1(t), p_1(t))} p_1(t) + \frac{(p(t), p_2(t))}{(p_2(t), p_2(t))} p_2(t) \\
&= \frac{34}{10} p_1(t) + 0 p_2(t) \\
&= \frac{17}{5} t
\end{aligned}
$$

□

3. (20 points) • *Find the equation $y = \beta_0 + \beta_1 x$ of the line that best fits the given data points, as a least squares model:*

$$\begin{bmatrix} x \\ y \end{bmatrix} : \quad \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

SOLUTION:

Solution. The best fit line is

$$\boxed{y = \frac{4}{5} + \frac{2}{5}x}$$

To find this, we have the matrices:

$$\mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}.$$

The best fit line is given by $\boldsymbol{\beta}$ satisfying

$$X^T X \boldsymbol{\beta} = X^T \mathbf{y}$$

or, since $\ker X = 0$ (which implies that $\ker X^T X = 0$),

$$\boldsymbol{\beta} = (X^T X)^{-1} X^T \mathbf{y}.$$

Using this latter formulation, we have

$$\boldsymbol{\beta} = \left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

$$= \left(\begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix} \right)^{-1} \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 16 \\ 8 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 2/5 \end{bmatrix}$$

□

4. • Consider the following real matrix

$$A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$$

(a) (4 points) *Find the characteristic polynomial $p_A(t)$ of A.*

SOLUTION:

Solution to (a). The characteristic polynomial of A is:

$$\boxed{p_A(t) = \det(tI - A) = t^3 - 11t^2 + 36t - 36}$$

If you used the textbook's convention, you will get $p_A(t) = \det(A - tI) = 36 - 36t + 11t^2 - t^3$; that is also fine.

Here is the computation.

$$\det(tI - A) = \begin{vmatrix} t-3 & +1 & -1 \\ +1 & t-5 & +1 \\ -1 & +1 & t-3 \end{vmatrix}$$

$$\begin{aligned}
&= (t-3)[(t-5)(t-3) - (1)(1)] - (1)[(t-3) - (1)(-1)] + (-1)[(1)(1) - (t-5)(-1)] \\
&= (t-3)[t^2 - 8t + 15 - 1] - [t - 3 + 1] - [1 + t - 5] \\
&= (t-3)[t^2 - 8t + 14] - [t-2] - [t-4] \\
&= [t^3 - 8t^2 + 14t - 3t^2 + 24t - 42] - 2t + 6 \\
&= t^3 - 11t^2 + 36t - 36
\end{aligned}$$

□

(b) (4 points) *Find the eigenvalues of A.*

SOLUTION:

Solution to (b). The eigenvalues of A are

$$\boxed{\lambda = 6, 3, 2}$$

The computation is as follows. By trying, $0, \pm 1, \pm 2$, we see that $p_A(2) = 0$. Thus we have

$$\begin{aligned} p_A(t) &= t^3 - 11t^2 + 36t - 36 \\ &= (t-2)(t^2 - 9t + 18) \\ &= (t-2)(t-3)(t-6) \end{aligned}$$

Therefore, the real roots of $p_A(t)$ are $\lambda = 6, 3, 2$. □

(c) (4 points) *Find a basis for each eigenspace of A in $\mathbb{R}^3$.*

SOLUTION:

Solution to (c). A basis for each eigenspace is:

$$\boxed{E_6 \leftrightarrow \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \quad E_3 \leftrightarrow \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \quad E_2 \leftrightarrow \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.}$$

The computation is as follows. We start with E_6. We want to find a basis for the kernel of

$$6I - A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

We put the matrix in reduced row echelon form:

$$\begin{bmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 3 \\ 3 & 1 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & -2 & -4 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

We then modify the matrix:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ \color{blue}{0} & \color{blue}{0} & \color{red}{-1} \end{bmatrix}$$

The last column, with the new red $\color{red}{-1}$, gives the basis element we want.

Next we consider E_3. We want to find a basis for the kernel of

$$3I - A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -2 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

We put the matrix in reduced row echelon form:

$$\begin{bmatrix} 0 & 1 & -1 \\ 1 & -2 & 1 \\ -1 & 1 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

We then modify the matrix:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ \color{blue}{0} & \color{blue}{0} & \color{red}{-1} \end{bmatrix}$$

The last column, with the new red $\color{red}{-1}$, gives the basis element we want.

Finally we consider E_2. We want to find a basis for the kernel of

$$2I - A = \begin{bmatrix} -1 & 1 & -1 \\ 1 & -3 & 1 \\ -1 & 1 & -1 \end{bmatrix}$$

We put the matrix in reduced row echelon form:

$$\begin{bmatrix} -1 & 1 & -1 \\ 1 & -3 & 1 \\ -1 & 1 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -1 & 1 \\ 1 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We then modify the matrix:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ \color{blue}{0} & \color{blue}{0} & \color{red}{-1} \end{bmatrix}$$

The last column, with the new red $\color{red}{-1}$, gives the basis element we want. $\square$

(d) (4 points) *Is A diagonalizable? If so, find a matrix $S \in M_{3\times3}(\mathbb{R})$ so that $S^{-1}AS$ is diagonal. If not, explain.*

SOLUTION:

Solution to (d). $\boxed{\text{Yes, } A \text{ is diagonalizable,}}$ since every symmetric matrix is diagonalizable.

We can use the matrix with columns given by the basis elements for the eigenspaces that we just computed. In other words, we may take

$$\boxed{S = \begin{bmatrix} -1 & -1 & 1 \\ 2 & -1 & 0 \\ -1 & -1 & -1 \end{bmatrix}.}$$

$\square$

(e) (4 points) *Is A diagonalizable with orthogonal matrices? If so, find an orthogonal matrix $U \in M_{3\times3}(\mathbb{R})$ so that $U^T AU$ is diagonal. If not, explain.*

SOLUTION:

Solution to (e). $\boxed{\text{Yes, } A \text{ is diagonalizable with orthogonal matrices,}}$ since every symmetric matrix is diagonalizable with orthogonal matrices.

We can use the matrix with columns given by the orthonormalized basis elements for the eigenspaces that we just computed (i.e., obtained by applying Gram–Schmidt to each basis for each eigenspace). Since we only have one dimensional eigenspaces, the Gram–Schmidt process simply divides each basis vector by its length, and so we may take

$$\boxed{U = \begin{bmatrix} -1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} \\ 2/\sqrt{6} & -1/\sqrt{3} & 0 \\ -1/\sqrt{6} & -1/\sqrt{3} & -1/\sqrt{2} \end{bmatrix}.}$$

$\square$

5. (20 points) • *Maximize the quadratic form*

$$Q(x_1, x_2, x_3) = 3x_1^2 - 2x_1x_2 + 2x_1x_3 + 5x_2^2 - 2x_2x_3 + 3x_3^2$$

subject to the constraint that $x_1^2 + x_2^2 + x_3^2 = 1$. [*Hint:* Compare to the matrix in Problem 3.]

SOLUTION:

Solution. The maximum of $Q(x_1, x_2, x_3)$ subject to the constraint $x_1^2 + x_2^2 + x_3^2 = 1$ is $\boxed{6}$.

To show this, one can see that for $\mathbf{x} = (x_1, x_2, x_3)$, we have

$$Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}, \qquad \text{where } A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

We already worked out in Problem 3 that the eigenvalues of A are $\lambda = 6, 3, 2$. Therefore the maximum of $Q(x_1, x_2, x_3)$ subject to the condition $x_1^2 + x_2^2 + x_3^2 = 1$ is 6, i.e., the largest eigenvalue of A. □

Although it is not asked for in this problem, let us note here that this maximum is achieved at the vectors $\pm\mathbf{u}_1 = \pm\frac{1}{\sqrt{6}}(-1, 2, -1)$. Indeed, we already worked out in Problem 3 that an orthonormal basis for each eigenspace of A is given by:

$$E_6 \leftrightarrow \mathbf{u}_1 = \frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \quad E_3 \leftrightarrow \mathbf{u}_2 = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \quad E_2 \leftrightarrow \mathbf{u}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Since the maximum of $Q(x_1, x_2, x_3)$ subject to the condition $x_1^2 + x_2^2 + x_3^2 = 1$ is achieved at any unit vector in the eigenspace for A with the largest eigenvalue (i.e., $\lambda = 6$), the maximum occurs at plus and minus the given orthonormal basis vector $\mathbf{u}_1$ for E_6 (since $\dim E_6 = 1$, the vectors $\pm\mathbf{u}_1$ are the only unit vectors in E_6).

6. (20 points) $\bullet$ *Find a singular value decomposition (SVD) of the matrix* $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}$.

SOLUTION:

Solution. An SVD for A is given by:

$$\boxed{A = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{18}} & -\frac{2}{3} \\ 0 & \frac{4}{\sqrt{18}} & -\frac{1}{3} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 5 & 0 \\ 0 & 3 \\ \hline 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}}_{V^T}}$$

To see this, we have

$$A^TA = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix},$$

which has characteristic polynomial

$$p_{A^TA}(t) = t^2 - (17+17)t + (17 \cdot 17 - 8 \cdot 8) = t^2 - 34t + 225 = (t-9)(t-25),$$

so that the eigenvalues of A^TA are $\lambda = 25, 9$, and the singular values of A are $\sigma = 5, 3$. We compute bases of each eigenspace of A^TA:

$$E_{25} = \ker(A^TA - 25I) = \ker \begin{bmatrix} 17-25 & 8 \\ 8 & 17-25 \end{bmatrix} = \ker \begin{bmatrix} -8 & 8 \\ 8 & -8 \end{bmatrix} \mapsto \underbrace{\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}}_{\text{RREF}} \mapsto \underbrace{\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}}_{\text{modified}}$$

so that $\mathbf{v}_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, or equivalently, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a basis for E_{25}.

Similarly:

$$E_9 = \ker(A^TA - 9I) = \ker \begin{bmatrix} 17-9 & 8 \\ 8 & 17-9 \end{bmatrix} = \ker \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix} \mapsto \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}_{\text{RREF}} \mapsto \underbrace{\begin{bmatrix} 1 & 1 \\ \color{blue}{0} & \color{red}{-1} \end{bmatrix}}_{\text{modified}}$$

so that $\mathbf{v}_2 = \begin{bmatrix} 1 \\ \color{red}{-1} \end{bmatrix}$ is a basis for E_9.

We then set

$$\mathbf{u}_1 := A\mathbf{v}_1 = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 := A\mathbf{v}_2 = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix},$$

(note that scaling $\mathbf{u}_i$ by a positive number is fine, but scaling by a negative number will lead to a sign error if you do not also change the sign of $\mathbf{v}_i$) and then extend to an orthogonal basis of $\mathbb{R}^3$ arbitrarily; I choose

$$\mathbf{u}_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}.$$

(In general, one extends to a basis, and then performs Gram–Schmidt.)

Therefore, dividing each of the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ by their length, and then setting V to be the matrix with columns given by the new unit length $\mathbf{v}_i$, setting U to be the matrix with columns given by the new unit length $\mathbf{u}_i$, and setting Σ to be the matrix with the singular values, we have $U\Sigma = AV$, which gives an SVD for A, namely $A = U\Sigma V^T$:

$$\underbrace{\begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{18}} & -\frac{2}{3} \\ 0 & \frac{4}{\sqrt{18}} & -\frac{1}{3} \end{bmatrix}}_{U} \underbrace{\left[\begin{array}{cc} 5 & 0 \\ 0 & 3 \\ \hline 0 & 0 \end{array}\right]}_{\Sigma} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}}_{V^T}$$

$\square$

7. • **TRUE** or **FALSE**. For this problem, and this problem only, **you do <u>not</u> need to justify your answer**.

(a) (2 points) **TRUE** or **FALSE** (circle one). *If* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, *then* $|\mathbf{x}.\mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$.

SOLUTION: TRUE: This is Cauchy–Schwarz.

(b) (2 points) **TRUE** or **FALSE** (circle one). *Two vectors in* $\mathbb{R}^n$ *are orthogonal if their dot product is zero.*

SOLUTION: TRUE: This was our definition of orthogonal.

(c) (2 points) **TRUE** or **FALSE** (circle one). *If* $W \subseteq \mathbb{R}^n$ *is a vector subspace and* $W^\perp$ *is the orthogonal complement, then* $W \subseteq W^\perp$.

SOLUTION: FALSE: For instance, take $n > 0$ and take $W = \mathbb{R}^n$, so that $W^\perp = 0$.

(d) (2 points) **TRUE** or **FALSE** (circle one). *If* $A \in \mathrm{M}_{m\times n}(\mathbb{R})$ *and* $\mathbf{b} \in \mathbb{R}^m$, *then a least squares solution to the equation* $A\mathbf{x} = \mathbf{b}$ *is a vector* $\hat{\mathbf{x}} \in \mathbb{R}^n$ *such that* $A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$.

SOLUTION: TRUE: We showed this in class – this is Theorem 13, p.363 of Lay.

(e) (2 points) **TRUE** or **FALSE** (circle one). *For the real vector space* $C^0([0,1])$ *consisting of continuous functions* $f : [0,1] \to \mathbb{R}$ *on the closed interval* $[0,1]$, *the rule*

$$(f(t), g(t)) = \int_0^1 f(t)g(t)\, dt$$

defines an inner product on $C^0([0,1])$.

SOLUTION: TRUE: We showed this in class, and this is also discussed in §6.7 of Lay.

(f) (2 points) **TRUE** or **FALSE** (circle one). *If A is any real matrix, then the matrix $A^T A$ has non-negative eigenvalues.*

SOLUTION: TRUE: We showed this in class. As a reminder, here is the sketch of the proof. Considering an eigenvector $\mathbf{x}$ for $A^T A$ with eigenvalue λ, one has $0 \leq \|A\mathbf{x}\|^2 = (A\mathbf{x})^T(A\mathbf{x}) = \mathbf{x}^T A^T A\mathbf{x} = \mathbf{x}^T \lambda \mathbf{x} = \lambda \|\mathbf{x}\|^2$. Dividing by $\|\mathbf{x}\|^2 > 0$ gives the assertion.

(g) (2 points) **TRUE** or **FALSE** (circle one). *Every real square matrix is diagonalizable with orthogonal matrices.*

SOLUTION: FALSE: There are some matrices that are not diagonalizable at all; e.g., $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(h) (2 points) **TRUE** or **FALSE** (circle one). *Given symmetric matrices A and B of the same size, then AB is a symmetric matrix.*

SOLUTION: FALSE: For instance, $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$.

(i) (2 points) **TRUE** or **FALSE** (circle one). *Every quadratic form has a maximum value.*

SOLUTION: FALSE: Take $Q(x) = x^2$. This quadratic form has no maximum value.

(j) (2 points) **TRUE** or **FALSE** (circle one). *Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then the angle θ between $\mathbf{x}$ and $\mathbf{y}$ satisfies* $\cos\theta = \dfrac{\mathbf{x}.\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}$.

SOLUTION: TRUE: This was our definition of the angle between vectors in $\mathbb{R}^n$.
