Final Exam

Linear Algebra: Matrix Methods

MATH 2130

Fall 2022

Sunday December 11, 2022

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NAME: ______________________________

PRACTICE EXAM

Question:	**1**	**2**	**3**	**4**	**5**	**6**	**7**	Total
Points:	20	20	20	20	20	20	20	140
Score:								

- The exam is closed book. You **may not use any resources** whatsoever, other than paper, pencil, and pen, to complete this exam.
- You **may not discuss the exam** with anyone except me, in any way, under any circumstances.
- You **must explain your answers**, and you will be **graded on the clarity of your solutions**.
- You must upload your exam as a single **.pdf** to **Canvas**, with the questions in the correct order, etc.
- You have 70 minutes to complete the exam.

1. (20 points) • Let $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{x}_4 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$.

Use the Gram–Schmidt process to find an orthonormal basis for the vector subspace of $\mathbb{R}^4$ *spanned by the vectors* $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, *and* $\mathbf{x}_4$.

2. (20 points) • Let $\mathbb{P}_3$ be the real vector space of polynomials of degree at most 3 (my notation for this vector space has been $\mathbb{R}[t]_3$, but here I am using the textbook's notation). A basis of $\mathbb{P}_3$ is given by the polynomials $1, t, t^2, t^3$.

We have seen that there is an inner product on $\mathbb{P}_3$ given by evaluation at -2, -1, 1, and 2. In other words, given polynomials $p(t), q(t) \in \mathbb{P}_3$, we define the inner product by the rule

$$\begin{aligned}(p(t), q(t)) &:= (p(-2), p(-1), p(1), p(2)).(q(-2), q(-1), q(1), q(2)) \\ &= p(-2)q(-2) + p(-1)q(-1) + p(1)q(1) + p(2)q(2).\end{aligned}$$

Let $p_1(t) = t$, and $p_2(t) = t^2$.

Find the best approximation to $p(t) = t^3$ *by the polynomials in* $\mathrm{Span}\{p_1(t), p_2(t)\}$.

In other words, find the polynomial $q(t)$ in the span of $p_1(t)$ and $p_2(t)$, that is closest to the polynomial $p(t)$ with respect to the given inner product on $\mathbb{P}_3$.

3. (20 points) • *Find the equation* $y = \boldsymbol{\beta}_0 + \boldsymbol{\beta}_1 x$ *of the line that best fits the given data points, as a least squares model:*

$$\begin{bmatrix} x \\ y \end{bmatrix} : \quad \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

4. • Consider the following real matrix

$$A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$$

(a) (4 points) *Find the characteristic polynomial* $p_A(t)$ *of* A.

(b) (4 points) *Find the eigenvalues of A.*

(c) (4 points) *Find a basis for each eigenspace of A in $\mathbb{R}^3$.*

(d) (4 points) *Is A diagonalizable? If so, find a matrix $S \in \mathrm{M}_{3\times 3}(\mathbb{R})$ so that $S^{-1}AS$ is diagonal. If not, explain.*

(e) (4 points) *Is A diagonalizable with orthogonal matrices? If so, find an orthogonal matrix $U \in \mathrm{M}_{3\times 3}(\mathbb{R})$ so that $U^T AU$ is diagonal. If not, explain.*

5. (20 points) • *Maximize the quadratic form*

$$Q(x_1, x_2, x_3) = 3x_1^2 - 2x_1x_2 + 2x_1x_3 + 5x_2^2 - 2x_2x_3 + 3x_3^2$$

subject to the constraint that $x_1^2 + x_2^2 + x_3^2 = 1$. [*Hint:* Compare to the matrix in Problem 3.]

6. (20 points) • *Find a singular value decomposition (SVD) of the matrix* $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}$.

7. • **TRUE** or **FALSE**. For this problem, and this problem only, **you do <u>not</u> need to justify your answer**.

(a) (2 points) **TRUE** or **FALSE** (circle one). *If* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, *then* $|\mathbf{x}.\mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$.

(b) (2 points) **TRUE** or **FALSE** (circle one). *Two vectors in* $\mathbb{R}^n$ *are orthogonal if their dot product is zero.*

(c) (2 points) **TRUE** or **FALSE** (circle one). *If* $W \subseteq \mathbb{R}^n$ *is a vector subspace and* $W^\perp$ *is the orthogonal complement, then* $W \subseteq W^\perp$.

(d) (2 points) **TRUE** or **FALSE** (circle one). *If* $A \in \mathrm{M}_{m\times n}(\mathbb{R})$ *and* $\mathbf{b} \in \mathbb{R}^m$, *then a least squares solution to the equation* $A\mathbf{x} = \mathbf{b}$ *is a vector* $\hat{\mathbf{x}} \in \mathbb{R}^n$ *such that* $A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$.

(e) (2 points) **TRUE** or **FALSE** (circle one). *For the real vector space* $C^0([0,1])$ *consisting of continuous functions* $f : [0,1] \to \mathbb{R}$ *on the closed interval* $[0,1]$, *the rule*

$$(f(t), g(t)) = \int_0^1 f(t)g(t)\, dt$$

defines an inner product on $C^0([0,1])$.

(f) (2 points) **TRUE** or **FALSE** (circle one). *If* A *is any real matrix, then the matrix* $A^T A$ *has non-negative eigenvalues.*

(g) (2 points) **TRUE** or **FALSE** (circle one). *Every real square matrix is diagonalizable with orthogonal matrices.*

(h) (2 points) **TRUE** or **FALSE** (circle one). *Given symmetric matrices* A *and* B *of the same size, then* AB *is a symmetric matrix.*

(i) (2 points) **TRUE** or **FALSE** (circle one). *Every quadratic form has a maximum value.*

(j) (2 points) **TRUE** or **FALSE** (circle one). *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. *Then the angle* θ *between* $\mathbf{x}$ *and* $\mathbf{y}$ *satisfies* $\cos\theta = \dfrac{\mathbf{x}.\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}$.