Reduced row echelon form of a matrix

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Introduction

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Putting a matrix in Reduced Row Echelon Form will be the main computational tool we will use in this class.

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- Every leading coefficient is 1 and is the only nonzero entry in its column.

$$\begin{bmatrix} \mathbf{1} & 3 & 0 & 0 & 2 \\ 0 & 0 & \mathbf{1} & 0 & 1 \\ 0 & 0 & 0 & \mathbf{1} & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Figure: A matrix in reduced row echelon form

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- B is obtained from A by adding a scalar multiple of one row of A to another.

We say that B is obtained from A by elementary row operations if there is a finite sequence of matrices $A = A_0, A_1, \ldots, A_n = B$, with A_{i+1} obtained from A_i, $i = 1, \ldots, n-1$, by an elementary row operation.

Example (Elementary Row Operations)

$$\begin{bmatrix} 3 & 9 & 27 & -3 \\ -3 & -11 & -35 & 5 \\ 2 & 8 & 26 & -4 \end{bmatrix} \quad A = A_0$$

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$$R_2' = R_3 + R_2 \quad \begin{bmatrix} 1 & 3 & 9 & -1 \\ -1 & -3 & -9 & 1 \\ 2 & 8 & 26 & -4 \end{bmatrix} \quad A_2 = B$$

Main Theorem

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Given a matrix A, there is a unique matrix $RREF(A)$ that is in Reduced Row Echelon Form that can be obtained from A by elementary row operations.

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Exercise. $\square$

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Example (Putting a matrix in RREF)

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$$
\begin{array}{c}
\\
\\
\\
R_1' = \frac{1}{3}R_1 \\
\\
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\end{array}
\begin{array}{c}
\left[\begin{array}{rrrr} 3 & 9 & 27 & -3 \\ -3 & -11 & -35 & 5 \\ 2 & 8 & 26 & -4 \end{array}\right] \\
\left[\begin{array}{rrrr} 1 & 3 & 9 & -1 \\ -3 & -11 & -35 & 5 \\ 1 & 4 & 13 & -2 \end{array}\right]
\end{array}
$$

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$$\begin{array}{r} \\ \\ \\ R_1' = \frac{1}{3}R_1 \\ \\ R_3' = \frac{1}{2}R_3 \\ \\ R_2' = 3R_1 + R_2 \\ R_3' = -R_1 + R_3 \end{array}
\begin{array}{c}
\begin{bmatrix} 3 & 9 & 27 & -3 \\ -3 & -11 & -35 & 5 \\ 2 & 8 & 26 & -4 \end{bmatrix} \\
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\begin{bmatrix} 1 & 3 & 9 & -1 \\ 0 & -2 & -8 & 2 \\ 0 & 1 & 4 & -1 \end{bmatrix}
\end{array}$$

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$$\begin{array}{l} R_2' = R_3 \\ R_3' = R_2 \end{array} \mapsto \begin{array}{r} \\ \\ R_3'' = 2R_2' + R_3' \end{array} \begin{bmatrix} 1 & 3 & 9 & -1 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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$$\begin{bmatrix} 3 & 9 & 27 & -3 \\ -3 & -11 & -35 & 5 \\ 2 & 8 & 26 & -4 \end{bmatrix}$$

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$$\begin{matrix} R_2' = R_3 \\ R_3' = R_2' \end{matrix} \mapsto \quad R_3'' = 2R_2 + R_3 \quad \begin{bmatrix} 1 & 3 & 9 & -1 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_1' = R_1 - 3R_2 \quad \begin{bmatrix} \mathbf{1} & 0 & -3 & 2 \\ 0 & \mathbf{1} & 4 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$