Exercise 7.5.3

Linear Algebra
MATH 2130

SEBASTIAN CASALAINA

ABSTRACT. This is Exercise 7.5.3 from Lay [LLM16, §7.5]:

Exercise 7.5.3. Consider the matrix of observations

$$\begin{bmatrix} 19 & 22 & 6 & 3 & 2 & 20 \\ 12 & 6 & 9 & 15 & 13 & 5 \end{bmatrix}$$

Find the principal components of the data.

Solution. The principal components of the data are:

$$\boxed{\mathbf{u}_1 \approx \begin{bmatrix} 0.95 \\ -0.32 \end{bmatrix} \text{ for } \lambda_1 \approx 95.2 \text{ and } \mathbf{u}_2 \approx \begin{bmatrix} 0.32 \\ 0.95 \end{bmatrix} \text{ for } \lambda_2 \approx 6.8}$$

We have the data matrix :

$$\mathbf{x} = \begin{bmatrix} 19 & 22 & 6 & 3 & 2 & 20 \\ 12 & 6 & 9 & 15 & 13 & 5 \end{bmatrix}$$

The sum of the elements in the first row is 72, and the sum of the elements in the second row is 60, so that the sample mean of the first row is 12, and the sample mean of the second row is 10. In other words, $\bar{\mathbf{x}} = \begin{bmatrix} 12 \\ 10 \end{bmatrix}$, so that the mean deviation form is:

$$\begin{aligned} \hat{\mathbf{x}} &= \begin{bmatrix} 19 & 22 & 6 & 3 & 2 & 20 \\ 12 & 6 & 9 & 15 & 13 & 5 \end{bmatrix} - \begin{bmatrix} 12 & 12 & 12 & 12 & 12 & 12 \\ 10 & 10 & 10 & 10 & 10 & 10 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 10 & -6 & -9 & -10 & 8 \\ 2 & -4 & -1 & 5 & 3 & -5 \end{bmatrix} \end{aligned}$$

Date: December 7, 2022.

The sample covariance matrix is then given by

$$\mathbf{s} = \frac{1}{6-1}\hat{\mathbf{x}}\hat{\mathbf{x}}^T = \frac{1}{5}\begin{bmatrix} 7 & 10 & -6 & -9 & -10 & 8 \\ 2 & -4 & -1 & 5 & 3 & -5 \end{bmatrix}\begin{bmatrix} 7 & 2 \\ 10 & -4 \\ -6 & -1 \\ -9 & 5 \\ -10 & 3 \\ 8 & -5 \end{bmatrix} = \begin{bmatrix} 86 & -27 \\ -27 & 16 \end{bmatrix}$$

Our goal is to find an orthonormal basis of eigenvectors for the sample covariance matrix $\mathbf{s}$. To this end, we have

$$p_{\mathbf{s}}(t) = t^2 - (86+16)t + (86 \cdot 16 - 27^2) = t^2 - 102 + 647.$$

This gives the eigenvalues of $\mathbf{s}$ as being $\lambda = 51 \pm \sqrt{1954} \approx 51 \pm 44.2041$. In other words,

$$\lambda_1 \approx 95.2041 \geq \lambda_2 \approx 6.79593$$

We now need to find bases for the eigenspaces E_{λ_1} and E_{λ_2}. We have

$$E_{\lambda_1} = \ker(\mathbf{s} - \lambda_1 I) \approx \begin{bmatrix} -9.20407 & -27 \\ -27 & -79.2041 \end{bmatrix} \mapsto \underbrace{\begin{bmatrix} 1 & 2.93348 \\ 0 & 0 \end{bmatrix}}_{\text{RREF}} \mapsto \underbrace{\begin{bmatrix} 1 & 2.93348 \\ 0 & -1 \end{bmatrix}}_{\text{modified}}$$

In other words a basis for E_{λ_1} is given by $\mathbf{v}_1 \approx \begin{bmatrix} 2.93348 \\ -1 \end{bmatrix}$. We have $\|\mathbf{v}_1\|^2 \approx 9.60530$, so that $\|\mathbf{v}_1\| \approx 3.09924$. Therefore an orthonormal basis for E_{λ_1} is given by

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} \approx \begin{bmatrix} 0.946516 \\ -0.322659 \end{bmatrix} \approx \begin{bmatrix} 0.95 \\ -0.32 \end{bmatrix}$$

Similarly, we have

$$E_{\lambda_2} = \ker(\mathbf{s} - \lambda_2 I) \approx \begin{bmatrix} 79.2041 & -27 \\ -27 & 9.20407 \end{bmatrix} \mapsto \underbrace{\begin{bmatrix} 1 & -0.340892 \\ 0 & 0 \end{bmatrix}}_{\text{RREF}} \mapsto \underbrace{\begin{bmatrix} 1 & -0.340892 \\ 0 & -1 \end{bmatrix}}_{\text{modified}}$$

In other words a basis for E_{λ_2} is given by $\mathbf{v}_2 \approx \begin{bmatrix} -0.340892 \\ -1 \end{bmatrix}$. We have $\|\mathbf{v}_2\|^2 \approx 1.11621$, so that $\|\mathbf{v}_2\| \approx 1.05651$. Therefore an orthonormal basis for E_{λ_2} is given by

$$\mathbf{u}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_1\|} \approx \begin{bmatrix} -0.322659 \\ -0.946516 \end{bmatrix} \approx \begin{bmatrix} -0.32 \\ -0.95 \end{bmatrix}$$

We are free to replace our basis vector by its negative, which is what I did in the solution given at the start. $\square$

REFERENCES

[LLM16] David Lay, Stephen Lay, and Judi McDonald, *Linear Algebra and its Applications*, Fifth edition, Pearson, 2016.

UNIVERSITY OF COLORADO, DEPARTMENT OF MATHEMATICS, CAMPUS BOX 395, BOULDER, CO 80309

Email address: `casa@math.colorado.edu`