Exercise 7.4.18

Linear Algebra
MATH 2130

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ABSTRACT. This is Exercise 7.4.18 from Lay [LLM16, §7.4]:

Exercise 7.4.18. Suppose that A is square and invertible. Find a singular value decomposition of A^{-1}.

Solution. Suppose that A is square and invertible, and $A = U\Sigma V^T$ is an SVD of A, then an SVD of A^{-1} is given by

$$\boxed{A^{-1} = V\Sigma^{-1}U^T.}$$

Indeed, since A is a square matrix, it follows that U, V, and Σ are square matrices of the same size as A. *A priori* we would have Σ looking like the following square diagonal matrix:

$$\left[\begin{array}{ccc|c} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_r & \\ \hline & 0 & & 0 \end{array}\right]$$

where $\sigma_1 \geq \cdots \geq \sigma_r > 0$. However, since A is also assumed to be invertible, and $A = U\Sigma V^T$ is a product of three square matrices of the same size, the matrices U, Σ, and V^T are all invertible. In particular, Σ is invertible, so it cannot have any zero entries on the diagonal (the determinant must be nonzero). Therefore, assuming that A is an $n \times n$ matrix, we have

$$\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}, \quad \Sigma^{-1} = \begin{bmatrix} 1/\sigma_1 & & \\ & \ddots & \\ & & 1/\sigma_n \end{bmatrix}$$

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At the same time, since the columns of U and V are given by non-zero orthonormal vectors, we can conclude that U and V are orthogonal matrices. In other words, $UU^T = VV^T = I$. Therefore

$$A^{-1} = (U\Sigma V^T)^{-1} = V\Sigma^{-1}U^T.$$

□

REFERENCES

[LLM16] David Lay, Stephen Lay, and Judi McDonald, *Linear Algebra and its Applications*, Fifth edition, Pearson, 2016.

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