Exercise 7.3.13

Linear Algebra
MATH 2130

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ABSTRACT. This is Exercise 7.3.13 from Lay [LLM16, §7.3]:

Exercise 7.3.13. Let A be an $n \times n$ symmetric matrix, let M and m denote the maximum and minimum values of the quadratic form

$$Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x},$$

where $\mathbf{x}^T\mathbf{x} = 1$, and denote the corresponding unit eigenvectors by $\mathbf{u}_1$ and $\mathbf{u}_n$. Given any number t between M and m, the steps below show how to find a unit vector $\mathbf{x}$ such that $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = t$.

(1) *Show there exists a number* α *between* 0 *and* 1 *such that*

$$t = (1-\alpha)m + \alpha M.$$

(2) If $n > 1$, setting $\mathbf{x} = \sqrt{1-\alpha}\mathbf{u}_n + \sqrt{\alpha}\mathbf{u}_1$, *show that if* $\mathbf{x}^T\mathbf{x} = 1$ *and* $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = t$.

Note that if $n = 1$, then $m = M$, so that $t = m = M$, and so, setting $\mathbf{x} = \mathbf{u}_1$, we have $\mathbf{x}^T\mathbf{x} = 1$ and $Q(\mathbf{x}) = Q(\mathbf{u}_1) = m = M = t$.

Solution to part (1). If $m = M$, then clearly we have $t = m = M$ so that $t = (1-\alpha)m + \alpha M$ for all α between 0 and 1. So assume that $m < M$. Then we have $t = (1-\alpha)m + \alpha M \iff t = m + \alpha(M - m) \iff \alpha = \frac{t-m}{M-m}$. □

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Solution to part (2). Using part (1), choose α between 0 and 1 such that $t = (1-\alpha)m + \alpha M$, and set $\mathbf{x} = \sqrt{1-\alpha}\mathbf{u}_n + \sqrt{\alpha}\mathbf{u}_1$. Then we have

$$\begin{aligned}
\mathbf{x}^T\mathbf{x} &= (\sqrt{1-\alpha}\mathbf{u}_n + \sqrt{\alpha}\mathbf{u}_1)^T(\sqrt{1-\alpha}\mathbf{u}_n + \sqrt{\alpha}\mathbf{u}_1) \\
&= (\sqrt{1-\alpha}\mathbf{u}_n^T + \sqrt{\alpha}\mathbf{u}_1^T)(\sqrt{1-\alpha}\mathbf{u}_n + \sqrt{\alpha}\mathbf{u}_1) \\
&= (1-\alpha)\|\mathbf{u}_n\|^2 + \sqrt{1-\alpha}\sqrt{\alpha}\mathbf{u}_n^T\mathbf{u}_1 + \sqrt{\alpha}\sqrt{1-\alpha}\mathbf{u}_1^T\mathbf{u}_n + \alpha\|\mathbf{u}_1\|^2 \\
&= (1-\alpha)\|\mathbf{u}_n\|^2 + \alpha\|\mathbf{u}_1\|^2 \\
&= (1-\alpha) + \alpha = 1.
\end{aligned}$$

And we also have

$$\begin{aligned}
Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} &= (\sqrt{1-\alpha}\mathbf{u}_n + \sqrt{\alpha}\mathbf{u}_1)^T A(\sqrt{1-\alpha}\mathbf{u}_n + \sqrt{\alpha}\mathbf{u}_1) \\
&= (\sqrt{1-\alpha}\mathbf{u}_n^T + \sqrt{\alpha}\mathbf{u}_1^T)A(\sqrt{1-\alpha}\mathbf{u}_n + \sqrt{\alpha}\mathbf{u}_1) \\
&= (1-\alpha)\mathbf{u}_n^T A\mathbf{u}_n + \sqrt{1-\alpha}\sqrt{\alpha}\mathbf{u}_n^T A\mathbf{u}_1 + \sqrt{\alpha}\sqrt{1-\alpha}\mathbf{u}_1^T A\mathbf{u}_n + \alpha\mathbf{u}_1^T A\mathbf{u}_1 \\
&= (1-\alpha)M + \sqrt{1-\alpha}\sqrt{\alpha}\mathbf{u}_n^T(m\mathbf{u}_1) + \sqrt{\alpha}\sqrt{1-\alpha}\mathbf{u}_1^T(M\mathbf{u}_n) + \alpha m \\
&= (1-\alpha)M + \alpha m = t.
\end{aligned}$$

□

REFERENCES

[LLM16] David Lay, Stephen Lay, and Judi McDonald, *Linear Algebra and its Applications*, Fifth edition, Pearson, 2016.

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