Exercise 7.2.24

Linear Algebra
MATH 2130

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ABSTRACT. This is Exercise 7.2.24 from Lay [LLM16, §7.2]:

Exercise 7.2.24. Suppose that $Q(\mathbf{x})$ is the quadratic form associated to the symmetric matrix

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix};$$

in other words, $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$. Verify the following statements:

a. Q is positive definite if and only if $\det(A) > 0$ and $a > 0$.
b. Q is negative definite if and only if $\det(A) > 0$ and $a < 0$.
c. Q is indefinite if and only if $\det(A) < 0$.

Solution. Let λ_1 and λ_2 be the roots of the characteristic polynomial of A, which are real numbers (see e.g., [LLM16, Thm. 3, p.399]).

a. From [LLM16, Thm. 5, p.407], we have that Q is positive definite $\iff \lambda_1, \lambda_2 > 0$. We have $\lambda_1, \lambda_2 > 0 \iff \lambda_1\lambda_2 > 0$ and $\lambda_1 + \lambda_2 > 0$, since the first equality is equivalent to λ_1 and λ_2 having the same sign, and the second equality then implies that the sign must be positive. In other words, using [LLM16, Exe. 7.2.23] that $\det(A) = \lambda_1\lambda_2$ and $\text{tr}(A) = \lambda_1 + \lambda_2$, we have that Q is positive definite if and only if $\det(A) > 0$ and $\text{tr}(A) > 0$. Using our particular matrix above, we can express the determinant and trace as $ad - b^2$ and $a + d$, respectively. Thus Q is positive definite $\iff ad - b^2 > 0$ and $a + d > 0$. But $ad - b^2 > 0$ and $a + d > 0 \iff ad - b^2 > 0$ and $a > 0$, since the equation $ad - b^2 > 0$ is equivalent to $ad > b^2 \geq 0$, so that it implies $ad > 0$, meaning that the sign of a and d must be equal. In other words, given that $ad - b^2 > 0$, having $a > 0$ is equivalent to having $a + d > 0$. In summary, we have shown that Q is positive definite if and only if $\det(A) > 0$ and $a > 0$.

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b. This is very similar to the last part, and so I will write the proof more concisely, as follows:

$$\begin{aligned} Q \text{ is negative definite } &\iff \lambda_1, \lambda_2 < 0 \\ &\iff \lambda_1\lambda_2 > 0 \text{ and } \lambda_1 + \lambda_2 < 0 \\ &\iff \det(A) > 0 \text{ and } \operatorname{tr}(A) < 0 \\ &\iff ad - b^2 > 0 \text{ and } a + d < 0 \\ &\iff ad - b^2 > 0 \text{ and } a < 0 \\ &\iff \det(A) > 0 \text{ and } a < 0 \end{aligned}$$

c. This is also very similar to the last to parts. We have that Q is indefinite if and only if λ_1 and λ_2 have opposite signs, which is equivalent to $\lambda_1\lambda_2 < 0$. In other words, Q is indefinite if and only if $\det(A) < 0$.

□

Remark 0.1. One can use similar techniques to show that Q is positive semi-definite if and only if $\det(A) \geq 0$ and $a, d \geq 0$ (since $ad - b^2 \geq 0$ and $a + d \geq 0$ is equivalent to $ad - b^2 \geq 0$ and $a, d \geq 0$). Similarly, Q is negative semi-definite if and only if $\det(A) \geq 0$ and $a, d \leq 0$ (since $ad - b^2 \geq 0$ and $a + d \leq 0$ is equivalent to $ad - b^2 \geq 0$ and $a, d \leq 0$).

Remark 0.2. Note that the matrix $A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$ shows that $\det(A) \geq 0$ and $a \geq 0$ is not enough to imply that A is positive semi-definite. Similarly, the matrix $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ shows that $\det(A) \geq 0$ and $a \leq 0$ is not enough to imply that A is negative semi-definite.

References

[LLM16] David Lay, Stephen Lay, and Judi McDonald, *Linear Algebra and its Applications*, Fifth edition, Pearson, 2016.

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