Exercise 2.3.27

Linear Algebra
MATH 2130

SEBASTIAN CASALAINA

ABSTRACT. This is Exercise 2.3.27 from Lay [LLM16, §2.3]:

Exercise 2.3.27. Suppose that A and B are square matrices of the same size, and AB is invertible. Show that A and B are invertible. [*Hint*: There is a matrix W such that $ABW = I$. Why?]

Solution. If AB is invertible, then by definition, there exists a square matrix W, of the same size as A and B, such that $WAB = ABW = I$. Therefore B has a left inverse (namely WA) and A has a right inverse (namely BW). Since a left inverse is also a right inverse, and conversely [LLM16, Theorem 8 j. and k., p.114, p.105, Exercise 2.1.25] (see below), we have that A and B are invertible. □

The textbook does not do a particularly good job of explaining why a left inverse is also a right inverse, and conversely. Here is a useful theorem that should be added to [LLM16, Theorem 8, p.114]:

Theorem 0.1 (Left inverses are right inverses, and conversely)**.** *Suppose that* $A \in \mathrm{M}_{n\times n}(\mathbb{R})$.

(1) *If there exists a matrix* $B \in \mathrm{M}_{n\times n}(\mathbb{R})$ *such that* $BA = I$*, then* $AB = I$*. In particular,* A *is invertible and* $B = A^{-1}$.
(2) *If there exists a matrix* $B \in \mathrm{M}_{n\times n}(\mathbb{R})$ *such that* $AB = I$*, then* $BA = I$*. In particular,* A *is invertible and* $B = A^{-1}$.

Proof. (1) Suppose there exists a matrix $B \in \mathrm{M}_{n\times n}(\mathbb{R})$ such that $BA = I$. Then by [LLM16, Theorem 8 j. and k.] there exists a matrix $C \in \mathrm{M}_{n\times n}(\mathbb{R})$ such that $AC = I$. If we consider the product BAC, we have that $C = IC = (BA)C = B(AC) = BI = B$, so that $C = B$ (as an aside, the conclusion that $C = B$ is a special case of [LLM16, Exercise 2.1.25]). Therefore $AB = AC = I$. As, $AB = BA = I$, we see that B is an inverse to A, and since inverses are unique (see [LLM16, p.105]), we have that $B = A^{-1}$.

Date: September 15, 2022.

(2) The proof is similar, and left to you. □

REFERENCES

[LLM16] David Lay, Stephen Lay, and Judi McDonald, *Linear Algebra and its Applications*, Fifth edition, Pearson, 2016.

UNIVERSITY OF COLORADO, DEPARTMENT OF MATHEMATICS, CAMPUS BOX 395, BOULDER, CO 80309

Email address: casa@math.colorado.edu