Exercise 1.7.2

Linear Algebra
MATH 2130

SEBASTIAN CASALAINA

ABSTRACT. This is Exercise 1.7.2 from Lay [LLM16, §1.7]:

Exercise 1.7.2. Determine if the following vectors are linearly independent:

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 5 \\ -8 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix},$$

Solution. The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent. Indeed, by definition, they are linearly independent if and only if we have that for all $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, if $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = 0$, then $\alpha_1 = \alpha_2 = \alpha_3 = 0$. This is equivalent to asking that that matrix equation

$$\begin{bmatrix} 0 & 0 & -3 \\ 0 & 5 & 4 \\ 2 & -8 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

have as its only solution the vector

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

But this is true, since we have

$$RREF\left(\begin{bmatrix} 0 & 0 & -3 \\ 0 & 5 & 4 \\ 2 & -8 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

□

Date: September 11, 2022.

Remark 0.1. Alternatively, we have seen that the rows of a matrix A are linearly independent if and only if there are no non-zero rows in $RREF(A)$. Therefore, we could consider the matrix with rows given by $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, namely $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 5 & -8 \\ -3 & 4 & 1 \end{bmatrix}$, and then check that

$$RREF\left(\begin{bmatrix} 0 & 0 & 2 \\ 0 & 5 & -8 \\ -3 & 4 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since this has no non-zero rows, we can conclude that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent.

References

[LLM16] David Lay, Stephen Lay, and Judi McDonald, *Linear Algebra and its Applications*, Fifth edition, Pearson, 2016.

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