Midterm I

Abstract Algebra 1

MATH 3140

Fall 2021

Friday September 24, 2021

NAME: ______________________________

PRACTICE EXAM

Question:	**1**	**2**	**3**	**4**	**5**	Total
Points:	20	20	20	20	20	100
Score:						

- The exam is closed book. You **may not use any resources** whatsoever, other than paper, pencil, and pen, to complete this exam.
- You **may not discuss the exam** with anyone except me, in any way, under any circumstances.
- You **must explain your answers**, and you will be **graded on the clarity of your solutions**.
- Either write your solutions **directly on this exam** or write the solution to **each problem on a separate piece of paper**.
- You must upload your exam as a single **.pdf** to **Canvas**, with the questions in the correct order, etc.
- You have 50 minutes to complete the exam. **Do not forget to leave yourself time (at least 5 minutes) at the end to upload your exam.**

1. • Consider the following subset of real 2×2 matrices:

$$H := \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in \mathbb{R} \right\} \subseteq \mathrm{M}_2(\mathbb{R}).$$

(a) (10 points) *Show that matrix multiplication defines a binary operation on H.*

(b) (10 points) *Does the map (or "function") $\phi : H \to \mathbb{R}$, given by*

$$\phi\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\right) = a,$$

give an isomorphism of the binary structure $\langle H, \cdot \rangle$ (here $\cdot$ denotes matrix multiplication) with the binary structure $\langle \mathbb{R}, + \rangle$? Explain.

1
20 points

2. (20 points) • Suppose that $\langle G, * \rangle$ is a binary structure such that:

1. The binary operation $*$ is associative.
2. There exists a **left** identity element; i.e., there exists $e \in G$ such that for all $g \in G$, we have $e * g = g$.
3. **Left** inverses exist; i.e., for all $g \in G$, there exists $g^{-1} \in G$ such that $g^{-1} * g = e$.

Show that $\langle G, * \rangle$ is a group.

2
20 points

3. (20 points) • Let H be a subgroup of a group G. For $a, b \in G$, let $a \sim b$ if and only if $a^{-1}b \in H$. *Show that $\sim$ is an equivalence relation on G.*

3
10 points

4. (a) (10 points) • *In the group $\mathbb{Z}_{28}$, what is the order of the subgroup generated by the element* 18*?*

(b) (10 points) *How many generators are there for the group $\mathbb{Z}_{28}$?*

4
20 points

5. • **True or False**. For this problem, and this problem only, **you do not need to justify your answer**.

(a) (4 points) **True or False** (circle one). Every subgroup of a cyclic group is cyclic.

(b) (4 points) **True or False** (circle one). If H and H' are subgroups of a group G, then $H \cap H'$ is a subgroup of G.

(c) (4 points) **True or False** (circle one). If $*$ is an associative binary operation on a set S, then for all $a, b, c \in S$, we have $(a * b) * c = c * (a * b)$.

(d) (4 points) **True or False** (circle one). Every finite group of at most 3 elements is abelian.

(e) (4 points) **True or False** (circle one). Every subgroup of an infinite group is infinite.

5
20 points