Take-Home Final

Abstract Algebra 1

MATH 3140

Fall 2021

Sunday December 12, 2021

NAME: ____

PRACTICE EXAM

SOLUTIONS

Question:	**1**	**2**	**3**	**4**	Total
Points:	25	25	25	25	100
Score:					

- For the exam you may use **only the following resources**: our textbook, your lecture notes, my lecture notes, your homework, the pdfs linked from the course webpage:

  `http://math.colorado.edu/~casa/teaching/21fall/3140/hw.html`

  and the quizzes and midterms we have taken on Canvas.
- You **may not use any other resources** whatsoever.
- You **may not discuss the exam** with anyone except me, in any way, under any circumstances.
- You **must explain your answers**, and you will be **graded on the clarity of your solutions**.
- You must upload your exam to **Canvas** as a **single .pdf** file with the questions in the correct order.
- The exam is due at 12:00 PM (noon) December 12, 2021.

1. (25 points) • Let G be a group with center $Z(G)$. *Show that if $G/Z(G)$ is cyclic, then $Z(G) = G$.*

[*Hint:* Show first there exists $g \in G$ such that for any $g_1 \in G$, there is a $z_1 \in Z(G)$ and $n_1 \in \mathbb{Z}$ such that $g_1 = g^{n_1} z_1$. Then show for any $g_1, g_2 \in G$ that $g_1 g_2 = g_2 g_1$.]

SOLUTION

Solution. It suffices to show that G is abelian (from the definition of the center, it follows immediately that a group G is abelian if and only if $G = Z(G)$). To show G is abelian, we must show that given $g_1, g_2 \in G$, then

$$g_1 g_2 = g_2 g_1.$$

To begin, since the group $G/Z(G)$ is cyclic, it has a generator $gZ(G) \in G/Z(G)$ for some $g \in G$. It follows that there are integers n_1, n_2 such that

$$g_1 Z(G) = (gZ(G))^{n_1} = g^{n_1} Z(G) \quad \text{and} \quad g_2 Z(G) = (gZ(G))^{n_2} = g^{n_2} Z(G).$$

Equivalently, $(g^{n_1})^{-1} g_1, (g^{n_2})^{-1} g_2 \in Z(G)$. We can rewrite this by saying that there exists $z_1, z_2 \in Z(G)$ such that $(g^{n_1})^{-1} g_1 = z_1$ and $(g^{n_2})^{-1} g_2 = z_2$, or rather, $g_1 = g^{n_1} z_1$ and $g_2 = g^{n_2} z_2$. Then

$$g_1 g_2 = g^{n_1} z_1 g^{n_2} z_2 = g^{n_2} z_2 g^{n_1} z_1 = g_2 g_1$$

since by definition z_1, z_2 commute with all elements of G, and g commutes with itself. □

1
25 points

2. (25 points) • **True or False**: *There exist a ring R with unity $1 \neq 0$, a ring R' with unity $1' \neq 0'$, and homomorphism of rings $\phi : R \to R'$ such that $\phi(1) \neq 0'$ and $\phi(1) \neq 1'$.*

SOLUTION

Solution. **True**: Let R_1, R_2 be rings with unity not equal to zero. For instance, let $R_1 = \mathbb{Z}$ and $R_2 = \mathbb{Z}$. Then the map

$$\phi : R_1 \longrightarrow R_1 \times R_2$$

$$r_1 \mapsto (r_1, 0_{R_2})$$

is a homomorphism of rings. Note that $1_{R_1 \times R_2} = (1_{R_1}, 1_{R_2})$, and $0_{R_1 \times R_2} = (0_{R_1}, 0_{R_2})$. In particular, $\phi(1_{R_1}) = (1_{R_1}, 0_{R_2}) \neq 1_{R_1 \times R_2}, 0_{1_{R_1 \times R_2}}$. □

2
25 points

3. (25 points) • Let D be an integral domain, and suppose that for every descending chain of ideals in D

$$\cdots \subseteq I_4 \subseteq I_3 \subseteq I_2 \subseteq I_1 \subseteq D$$

there is a positive integer n such that $I_m = I_n$ for all $m \geq n$. *Show that D is a field.*

SOLUTION

Solution. Let $0 \neq x \in D$, and consider the chain of ideals

$$\cdots \subseteq (x^4) \subseteq (x^3) \subseteq (x^2) \subseteq (x) \subseteq D$$

Then there is some positive integer n such that $(x^{n+1}) = (x^n)$. In particular, $x^n \in (x^{n+1})$, so that by definition there exists $y \in D$ such that $x^n = yx^{n+1}$. In other words, $x^n - yx^{n+1} = 0$, or,

$$(1 - yx)x^n = 0.$$

Since we are in an integral domain, and $x \neq 0$, we have that $x^n \neq 0$, and finally that $1 - yx = 0$, so that $yx = 1$ and therefore x is a unit. Since we have shown that every nonzero element of D is a unit, we have that D is a field. □

3
25 points

4. (25 points) • *Show that if F, E, and K are fields with $F \leq E \leq K$, then K is algebraic over F if and only if E is algebraic over F, and K is algebraic over E.* (You must *not* assume the extensions are finite.)

SOLUTION

Solution. This is Fraleigh Exercise 31.31. The solution is available on the course webpage. □

4
25 points