Exercise 27.24

Abstract Algebra 1
MATH 3140

SEBASTIAN CASALAINA

ABSTRACT. This is Exercise 27.24 from Fraleigh [Fra03, §27]:

Exercise 27.24. Let R be a finite commutative ring with unity. Show that every prime ideal in R is a maximal ideal.

Solution. Let R be a finite commutative ring with unity 1. If $1 = 0$, then $R = \{0\}$, and so R contains no ideals $I \subsetneq$ properly contained in R, and therefore R contains no prime ideals. Therefore, trivially, we have that every prime ideal in R is maximal, since this is a vacuous statement.

So, assume now that $1 \neq 0$. If $\mathfrak{p} \subseteq R$ is a prime ideal in R, then from [Fra03, Theorem 27.16] we have that the quotient ring ("factor ring") $R/\mathfrak{p}$ is an integral domain. Since R is finite, and there is a surjective ("onto") ring homomorphism $R \to R/\mathfrak{p}$, it follows that $R/\mathfrak{p}$ is finite. In other words, $R/\mathfrak{p}$ is a finite integral domain. From [Fra03, Theorem 19.11], any finite integral domain is a field, and so we have that $R/\mathfrak{p}$ is a field. Finally, from [Fra03, Theorem 27.9], since $R/\mathfrak{p}$ is a field, it follows that $\mathfrak{p}$ is a maximal ideal. Thus every prime ideal in R is a maximal ideal. □

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References

[Fra03] John Fraleigh, *A First Course in Abstract Algebra*, Seventh edition, Addison Wesley, Pearson, 2003.

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