Exercise 27.2

Abstract Algebra 1
MATH 3140

SEBASTIAN CASALAINA

ABSTRACT. This is Exercise 27.2 from Fraleigh [Fra03, §27]:

Exercise 27.2. Find all prime ideals and all maximal ideals of $\mathbb{Z}_{12}$.

Solution. The ideals of $\mathbb{Z}_{12}$ are:

$$\langle 0\rangle = \{0\},\ \langle 1\rangle = \mathbb{Z}_{12},\ \langle 2\rangle,\ \langle 3\rangle,\ \langle 4\rangle,\ \langle 6\rangle.$$

One can confirm this by seeing that this is a full list of subgroups of $\mathbb{Z}_{12}$, and then checking that these are again ideals (they are the principal ideals generated by the indicated element; see [Fra03, Definition 27.21]).

Of these ideals, the prime ideals are

$$\langle 2\rangle,\ \langle 3\rangle.$$

Indeed, from [Fra03, Theorem 27.15], we need to check which factor rings are integral domains. We have $\mathbb{Z}_{12}/\langle 0\rangle = \mathbb{Z}_{12}$ is not an integral domain, we have $\mathbb{Z}_{12}/\langle 1\rangle = 0$ is not an integral domain, we have $\mathbb{Z}_{12}/\langle 4\rangle \cong \mathbb{Z}_4$ is not an integral domain, and we have $\mathbb{Z}_{12}/\langle 6\rangle \cong \mathbb{Z}_6$ is not an integral domain. On the other hand, $\mathbb{Z}_{12}/\langle 2\rangle \cong \mathbb{Z}_2$ is a field (which is an integral domain), and $\mathbb{Z}_{12}/\langle 3\rangle \cong \mathbb{Z}_3$ is a field.

Both of the prime ideals

$$\langle 2\rangle,\ \langle 3\rangle$$

are maximal ideals. Indeed, from [Fra03, Theorem 27.9], we need to check which factor rings are fields. But we have already done this in the previous paragraph. □

Remark 0.1. As an alternative approach, we can start by listing all the ideals in $\mathbb{Z}_{12}$. These are $\langle 0\rangle = \{0\}$, $\langle 1\rangle = \{0, 1, 2, \ldots, 12\}$, $\langle 2\rangle = \{0, 2, 4, 6, 8, 10\}$, $\langle 3\rangle = \{0, 3, 6, 9\}$, $\langle 4\rangle = \{0, 4, 8\}$, and

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$\langle 6\rangle = \{0, 6\}$. From this list we can see that the maximal ideals are $\langle 2\rangle$ and $\langle 3\rangle$ (they are the only proper ideals that have no other proper ideals containing them). All of the other proper ideals can be checked to not be prime directly: $3 \cdot 4 = 0 \in \langle 0\rangle$, but $3, 4 \notin \langle 0\rangle$, and similarly, $2 \cdot 2 = 4 \in \langle 4\rangle$, $2 \cdot 3 = 6 \in \langle 6\rangle$.

References

[Fra03] John Fraleigh, *A First Course in Abstract Algebra*, Seventh edition, Addison Wesley, Pearson, 2003.

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