Exercise 15.36

Abstract Algebra 1
MATH 3140

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ABSTRACT. This is Exercise 15.36 from Fraleigh [Fra03, §15]:

Exercise 15.36. Let $\phi : G \to G'$ be a group homomorphism, and let N' be a normal subgroup of G'. Show that $\phi^{-1}[N']$ is a normal subgroup of G.

Solution. Consider the homomorphism $\pi' : G' \to G'/N'$ (with $\ker \pi' = N'$), and the homomorphism obtained as the composition

$$G \xrightarrow{\phi} G' \xrightarrow{\pi'} G'/N'.$$

Since N' is the identity element of the group G'/N', we we have $\ker(\pi' \circ \phi) = (\pi' \circ \phi)^{-1}[N'] = \phi^{-1}[\pi'^{-1}[N']] = \phi^{-1}[N']$, since $\ker \pi' = \pi'^{-1}[N'] = N'$. Therefore, since $\phi^{-1}[N']$ is the kernel of a homomorphism (i.e., the kernel of $\pi' \circ \phi$), it is a normal subgroup [Fra03, Corollary 13.20, p.132]. □

Here is another solution:

Solution. Let $\phi : G \to G'$ be a group homomorphism, and let N' be a normal subgroup of G'. Recall from [Fra03, Theorem 13.12] that

$$\phi^{-1}[N'] := \{g \in G : \phi(g) \in N\}$$

is a subgroup of G. Therefore, from [Fra03, Theorem 14.13], to show that $\phi^{-1}[N]$ is a normal subgroup, it suffices to show that for all $h \in \phi^{-1}[N]$ and for all $g \in G$, we have $ghg^{-1} \in \phi^{-1}[N]$. From the definition of $\phi^{-1}[N]$, to show that $ghg^{-1} \in \phi^{-1}[N]$ we must show that $\phi(ghg^{-1}) \in N$. To

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this end, we have

$$\phi(ghg^{-1}) = \phi(g)\phi(h)\phi(g^{-1}) = \phi(g)\phi(h)\phi(g)^{-1}.$$

But we are assuming that $\phi(h) \in N$ (by assumption $h \in \phi^{-1}[N]$), and that N is a normal subgroup, so that using [Fra03, Theorem 14.13] again, we have that $\phi(g)\phi(h)\phi(g)^{-1} \in N$. Thus $\phi(ghg^{-1}) \in N$, and we are done. $\square$

REFERENCES

[Fra03] John Fraleigh, *A First Course in Abstract Algebra*, Seventh edition, Addison Wesley, Pearson, 2003.

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