# AXIOMS FOR INCIDENCE PLANES

## MATH 2001

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ABSTRACT. These are the axioms we will use for incidence planes. These are meant to clarify the meaning of the definition [Har00, p.66].

## 1. PLANES

**Definition 1.1** (Plane)**.** A *plane* consists of a pair $(\Pi, \Lambda)$ where $\Pi$ is a set and $\Lambda \subseteq \mathscr{P}(\Pi)$ is a subset of the power set of $\Pi$.

We will often call the elements of $\Pi$ *points*, and refer to $\Pi$ as the set of points in the plane. Similarly, we will often call the elements of $\Lambda$ *lines*, and refer to $\Lambda$ as the set of lines in the plane. We say *a point* $p \in \Pi$ *is contained in a line* $\ell \in \Lambda$ if $p \in \ell$.

**Example 1.2** (Empty example)**.** Let $\Pi = \emptyset$ and let $\Lambda = \emptyset$. Then $(\Pi, \Lambda)$ is a plane.

**Example 1.3** (Real Cartesian plane)**.** Let $\Pi = \mathbb{R}^2$. An affine linear subset of $\mathbb{R}^2$ is a subset $\ell \subseteq \mathbb{R}^2$ such there exist real numbers $a, b, c \in \mathbb{R}$ with $a \neq 0$ or $b \neq 0$, and

$$\ell = \{(x, y) \in \mathbb{R}^2 : ax + by + c = 0\}.$$

Let $\Lambda \subseteq \mathbb{P}(\Pi)$ be the set of affine linear subsets of $\mathbb{R}^2$. Then $(\Pi, \Lambda)$ is a plane, which we call the *real Cartesian plane*.

## 2. INCIDENCE PLANES

**Definition 2.1** (Incidence plane)**.** An *incidence plane* is a plane $(\Pi, \Lambda)$ satisfying the following conditions:

(I1) (Two distinct points determine a unique line) For every $p_1, p_2 \in \Pi$ with $p_1 \neq p_2$, there exists a unique $\ell \in \Lambda$ such that $p_1 \in \ell$ and $p_2 \in \ell$.

(I2) (Every line determines two distinct points) For every $\ell \in \Lambda$, there exist $p_1, p_2 \in \Pi$ with $p_1 \neq p_2$ such that $p_1 \in \ell$ and $p_2 \in \ell$.

(I3) (There exist 3 distinct non-colinear points) There exist $p_1, p_2, p_3 \in \Pi$ with $p_1 \neq p_2$, $p_1 \neq p_3$, and $p_2 \neq p_3$, such that there does not exist $\ell \in \Lambda$ with $p_1 \in \ell$, $p_2 \in \ell$ and $p_3 \in \ell$.

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**Exercise 2.2** (Incidence planes have points and lines)**.** *Let* $(\Pi, \Lambda)$ *be an incidence plane.*

(1) *Show that* $\Pi \neq \emptyset$*. In fact, show that* $\Pi$ *has at least* 3 *elements.*
(2) *Show that* $\Lambda \neq \emptyset$*. In fact, show that* $\Lambda$ *has at least* 3 *elements.*

**Exercise 2.3** (Lines contain points, but not all the points)**.** *Let* $(\Pi, \Lambda)$ *be an incidence plane.*

(1) *Show that* $\emptyset \notin \Lambda$*.*
(2) *Show that* $\Pi \notin \Lambda$*.*

**Example 2.4** (Minimal incidence plane)**.** Here we consider the case where

$$\Pi = \{1,2,3\} \text{ and } \Lambda = \{\{1,2\},\{1,3\},\{2,3\}\}.$$

Then $(\Pi, \Lambda)$ is an incidence plane. Indeed, first, since $\Lambda \subseteq \mathscr{P}(\Pi) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \Pi\}$, we have that $(\Pi, \Lambda)$ is a plane. To check $(\Pi, \Lambda)$ is an incidence plane, we need to check that it satisfies (I1), (I2), and (I3) above. For (I1), there are three possible choices for $p_1, p_2 \in \Pi$ with $p_1 \neq p_2$. We can have $p_1 = 1$ and $p_2 = 2$, or we can have $p_1 = 1$ and $p_2 = 3$, or we can have $p_1 = 2$ and $p_2 = 3$. In the first case, namely $p_1 = 1$ and $p_2 = 2$, then $\{1,2\} \in \Lambda$ is the only $\ell \in \Lambda$ with $p_1 = 1 \in \ell$ and $p_2 = 2 \in \ell$. The other cases are similar, and are left to you to confirm as an exercise. For (I2), there are three $\ell \in \Lambda$ to consider, namely $\{1,2\}$, $\{1,3\}$, and $\{2,3\}$. In the first case, namely $\ell = \{1,2\}$, then $1,2 \in \{1,2\}$ with $1 \neq 2$. The other cases are similar, and are left to you to confirm as an exercise. For (I3), we can take $p_1 = 1$, $p_2 = 2$, and $p_3 = 3$. We have $p_1 \neq p_2$, $p_1 \neq p_3$, and $p_2 \neq p_3$, and moreover, there is no $\ell \in \Lambda$ with $p_1 = 1 \in \ell$, $p_2 = 2 \in \ell$, and $p_3 = 3 \in \ell$. Thus we have confirmed that the plane $(\Pi, \Lambda)$ satisfies (I1), (I2), and (I3), so that $(\Pi, \Lambda)$ is an incidence plane.

**Example 2.5** (Real Cartesian plane)**.** Let $(\Pi, \Lambda)$ be the real Cartesian plane (Example 1.2). Then $(\Pi, \Lambda)$ is an incidence plane. We have already seen that $(\Pi, \Lambda)$ is a plane. To check it is an incidence plane, we need to confirm (I1), (I2), and (I3). For (I1), given $p_1 = (a_1, a_2) \in \mathbb{R}^2$ and $p_2 = (b_1, b_2) \in \mathbb{R}^2$, then $p_1$ and $p_2$ are contained in the line

$$y - a_2 = \frac{b_2 - a_2}{b_1 - a_1}(x - a_1)$$

if $a_1 \neq b_1$; if $a_1 = b_1$, they are contained in the line $x = a_1$. For (I2), if $\ell = \{ax + by + c = 0\}$ with $b \neq 0$, then we may take $p_1 = (0, -c/b)$, and $p_2 = (1, -(c+a)/b)$. If $b = 0$, then we may take $p_1 = (-c/a, 0)$ and $p_2 = (-c/a, 1)$. For (I3), we may take $p_1 = (0,0)$, $p_2 = (1,0)$, and $p_3 = (0,1)$; it is left to you as an exercise to show that there do not exist $a, b, c \in \mathbb{R}$ with $a \neq 0$ or $b \neq 0$, so that $p_1$, $p_2$, and $p_3$ all satisfy the equation $ax + by + c = 0$.

**Definition 2.6** (Parallel lines)**.** Given a plane $(\Pi, \Lambda)$, we say that $\ell_1, \ell_2 \in \Lambda$ are *parallel* if $\ell_1 \neq \ell_2$ and there is no $p \in \Pi$ such that $p \in \ell_1$ and $p \in \ell_2$. We also say that $\ell_1, \ell_2 \in \Lambda$ are parallel if $\ell_1 = \ell_2$.

## 3. SOME FURTHER EXERCISES

In the following exercises, we will assume that we are given an incidence plane $(\Pi, \Lambda)$.

**Exercise 3.1** (Non-parallel lines meet in a unique point)**.** *If $\ell_1, \ell_2 \in \Lambda$ are not parallel, then there is a unique $p \in \Pi$ with $p \in \ell_1$ and $p \in \ell_2$.*

**Exercise 3.2** (There exist three distinct lines not all meeting at one point)**.** *There exist $\ell_1, \ell_2, \ell_3 \in \Lambda$ with $\ell_1 \neq \ell_2$, $\ell_1 \neq \ell_3$, and $\ell_2 \neq \ell_3$, such that there does not exist $p \in \Pi$ with $p \in \ell_1$, $p \in \ell_2$, and $p \in \ell_3$.*

**Exercise 3.3** (There is at least one point not contained in any given line)**.** *Given $\ell \in \Lambda$, there exists $p \in \Pi$ with $p \notin \ell$.*

**Exercise 3.4** (There is at least one line not containing any given point)**.** *Given $p \in \Pi$, there exists $\ell \in \Lambda$ with $p \notin \ell$.*

**Exercise 3.5** (There are at least two distinct lines containing any given point)**.** *GIven $p \in \Pi$, there exist $\ell_1, \ell_2 \in \Lambda$ with $\ell_1 \neq \ell_2$ such that $p \in \ell_1$ and $p \in \ell_2$.*

## REFERENCES

[Har00] Robin Hartshorne, *Geometry: Euclid and beyond*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 2000. MR 1761093

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