FINAL EXAM
LINEAR ALGEBRA

MATH 2135

Monday May 7, 2018
1:30 PM – 3:30 PM

Name

PRACTICE EXAM

Please answer all of the questions, and show your work.
You must explain your answers to get credit.
You will be graded on the clarity of your exposition!

1	2	3	4	5	6	7	8	9	10	
10	10	10	10	10	10	10	10	10	10	10 total

Date: May 4, 2018.

1. *Give the definition of a vector space.*

1
10 points

2. Consider the following matrix

2
10 points

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & 1 & 3 \end{pmatrix}$$

2.(a). *Find the characteristic polynomial $p_A(t)$ of A.*

2.(b). *Find the eigenvalues of A.*

2.(c). *Find an orthonormal basis for each eigenspace of A in $\mathbb{C}^3$.*

2.(d). *Is A diagonalizable? If so, find a matrix $S \in \mathrm{M}_{3\times 3}(\mathbb{C})$ so that $S^{-1}AS$ is diagonal. If not, explain.*

2.(e). *Is A diagonalizable with unitary matrices? If so, find a unitary matrix $U \in \mathrm{M}_{3\times 3}(\mathbb{C})$ so that U^*AU is diagonal. If not, explain.*

3. Consider the following matrix

3
10 points

$$B = \begin{pmatrix} 1 & 2 & 0 & 2 & -1 & 0 \\ 3 & -1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 2 & 8 & 6 \\ 0 & 0 & 0 & 3 & -3 & 0 \end{pmatrix}$$

3.(a). *What is the sum of the roots of the characteristic polynomial of B?*

3.(b). *What is the product of the roots of the characteristic polynomial of B?*

3.(c). *Does B admit an orthonormal basis of eigenvectors in* $\mathbb{R}^6$*?*

4
10 points

4. Suppose that $(V_1, +_1, \cdot_1)$ and $(V_2, +_2, \cdot_2)$ are K-vector spaces.
Define maps:

$$+ : (V_1 \times V_2) \times (V_1 \times V_2) \to V_1 \times V_2$$
$$(v_1, v_2) + (v_1', v_2') = (v_1 +_1 v_1', v_2 +_2 v_2')$$

and

$$\cdot : K \times (V_1 \times V_2) \to V_1 \times V_2$$
$$\lambda \cdot (v_1, v_2) = (\lambda \cdot_1 v_1, \lambda \cdot_2 v_2).$$

Show that the triple $(V_1 \times V_2, +, \cdot)$ *is a* K*-vector space.*

We denote this vector space by $V_1 \times V_2$ or $V_1 \oplus V_2$, and call it the direct product, or direct sum, respectively, of the vector spaces V_1 and V_2.

5
10 points

5. Suppose that W is a finite dimensional subspace of a Euclidean space $(V, (-,-))$. Suppose that $L : V \to V$ is a linear map such that

- $\operatorname{Im} L = W$.
- $L \circ L = L$.
- $\ker L = W^\perp$.

Show that if $e_1, \ldots, e_n$ form an orthonormal basis for W, then L is given by

$$L(v) = \sum_{i=1}^{n} (v, e_i) e_i.$$

In other words, show that L is the orthogonal projection onto W.

6. Let $(V, (-,-))$ be a Euclidean space and suppose that $L : V \to V$ is a linear map admitting an adjoint L^*. *If $L = L^*$, show that all the eigenvalues of L are real.*

6
10 points

7. *Show that an $m \times n$ matrix, with $m \leq n$, has rank m if and only if it has an $m \times m$ minor with non-zero determinant.*

7
10 points

8. In this problem we will work with matrices $A, B \in \mathrm{M}_{n\times n}(\mathbb{C})$.

8
10 points

8.(a). We say that A is similar to B, and write $A \sim B$, if there is an invertible matrix $S \in \mathrm{M}_{n\times n}(\mathbb{C})$ such that $B = S^{-1}AS$. *Show that $\sim$ defines an equivalence relation on $\mathrm{M}_{n\times n}(\mathbb{C})$.*

8.(b). *Show that any $A \in \mathrm{M}_{n\times n}(\mathbb{C})$ is similar to an upper triangular matrix.*

8.(c). Suppose that $A \in \mathrm{M}_{2\times 2}(\mathbb{C})$ has distinct eigenvalues λ_1 and λ_2. *Show that A is similar to a diagonal matrix with λ_1 and λ_2 on the diagonal.*

8.(d). Suppose that $A \in \mathrm{M}_{2\times 2}(\mathbb{C})$ has a single eigenvlaue λ. *Show that A is similar to either λI or to a matrix of the form*

$$J_\lambda := \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

and that λI is not similar to J_λ.

8.(e). *Use the previous parts of the problem to describe the equivalence classes of matrices in $\mathrm{M}_{2\times 2}(\mathbb{C})$ under the equivalence relation $\sim$.*

9
10 points

9. In this problem we will work with matrices $A, B \in \mathrm{M}_{n\times n}(\mathbb{C})$.

9.(a). We say that A is unitarily similar to B, and write $A \sim_U B$ if there is a unitary matrix $U \in \mathrm{M}_{n\times n}(\mathbb{C})$ such that $B = U^*AU$. *Show that $\sim_U$ defines an equivalence relation on $\mathrm{M}_{n\times n}(\mathbb{C})$.*

9.(b). *Show that any $A \in \mathrm{M}_{n\times n}(\mathbb{C})$ is unitarily similar to an upper triangular matrix.*

9.(c). Suppose that $A, B \in \mathrm{M}_{n\times n}(\mathbb{C})$ are *upper triangular*, with the same diagonal entries $a_{ii} = b_{ii}$, $i = 1, \ldots, n$, with $a_{ii} \neq a_{jj}$, $i \neq j$. *If $U \in M_{n\times n}(\mathbb{C})$ is a unitary matrix such that $B = U^*AU$, then show that U is diagonal.*

9.(d). Suppose $T \in \mathrm{M}_{2\times 2}(\mathbb{C})$ is an upper triangular matrix:

$$T = \begin{pmatrix} \lambda_1 & t_{12} \\ 0 & \lambda_2 \end{pmatrix}.$$

Show that

$$|t_{12}|^2 = \mathrm{tr}(T^*T) - |\lambda_1|^2 - |\lambda_2|^2.$$

9.(e). Suppose that $A, B \in \mathrm{M}_{n\times n}(\mathbb{C})$ are unitarily similar. *Show that* $\mathrm{tr}(A^*A) = \mathrm{tr}(B^*B)$.

10. **TRUE** or **FALSE**. You do **not** need to justify your answer.

10
10 points

10.(a). Let $(V, (-,-))$ be a Euclidean space, and let $v, w \in V$. Then

$$||v + w||^2 = ||v||^2 + ||w||^2$$

if and only if v and w are orthogonal.

T F

10.(b). Suppose that $T : V \to V'$ is a linear map of finite dimensional vector spaces. Then $\dim V' = \dim \ker(T) + \dim \mathrm{Im}(T)$.

T F

10.(c). The cofactor matrix of an $n \times n$ matrix can only have rank equal to n, 1 or 0.

T F

10.(d). Suppose that $A \in \mathrm{M}_{n\times n}(\mathbb{R})$ is symmetric, and let $v_1, v_2 \in \mathbb{R}^n$ be eigenvectors with corresponding eigenvalues λ_1, λ_2. If $\lambda_1 \neq \lambda_2$, then v_1 is orthogonal to v_2.

T F

10.(e). The row space of a matrix is the same as the row space of the reduced row echelon form of the matrix.

T F

10.(f). Suppose that M is an $n \times n$ matrix and $M^N = 0$ for some integer $N > 1$. Then M is diagonalizable.

T F

10.(g). Let A be an $n \times n$ matrix. Then $p_A(A) = 0$.

T F

10.(h). Let $(V, (-,-))$ be a Euclidean space, and let $v, w \in V$. Then $|v.w| \leq ||v||||w||$.

T F

10.(i). An $n \times n$ matrix has n linearly independent eigenvectors if and only if it has n distinct eigenvalues.

T F

10.(j). Let $(V, (-,-))$ be a Euclidean space, and let $v, w \in V$. Then

$$||v + w|| \leq ||v|| + ||w||.$$

T F