CHAPTER 6

A brief introduction to linear algebra

1. Vector spaces and linear maps

In what follows, fix $K \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$. More generally, K can be any field.

1.1. Vector spaces. Motivated by our intuition of adding and scaling vectors in the plane (see Figure 1), we make the following definition:

Definition 6.1.1 (K-vector space)**.** *A K**-vector space** consists of a triple $(V, +, \cdot)$, where V is a set, and $+ : V \times V \to V$ and $\cdot : K \times V \to V$ are maps, satisfying the following properties:*

(1) *(Group laws)*
   (a) *(Additive identity) There exists an element $\mathscr{O} \in V$ such that for all $v \in V$, $v + \mathscr{O} = v$;*
   (b) *(Additive inverse) For each $v \in V$ there exists an element $-v \in V$ such that $v + (-v) = \mathscr{O}$;*
   (c) *(Associativity of addition) For all $v_1, v_2, v_3 \in V$,*
$$(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3);$$
(2) *(Abelian property)*
   (a) *(Commutativity of addition) For all $v_1, v_2 \in V$,*
$$v_1 + v_2 = v_2 + v_1;$$
(3) *(Module conditions)*
   (a) *For all $\lambda \in K$ and all $v_1, v_2 \in V$,*
$$\lambda \cdot (v_1 + v_2) = (\lambda \cdot v_1) + (\lambda \cdot v_2);$$
   (b) *For all $\lambda_1, \lambda_2 \in K$, and all $v \in V$,*
$$(\lambda_1 + \lambda_2) \cdot v = (\lambda_1 \cdot v) + (\lambda_2 \cdot v);$$
   (c) *For all $\lambda_1, \lambda_2 \in K$, and all $v \in V$,*
$$(\lambda_1 \lambda_2) \cdot v = \lambda_1 \cdot (\lambda_2 \cdot v);$$
   (d) *For all $v \in V$,*
$$1 \cdot v = v.$$

In the above, for all $\lambda \in K$ and all $v, v_1, v_2 \in V$ we have denoted $+(v_1, v_2)$ by $v_1 + v_2$ and $\cdot(\lambda, v)$ by $\lambda \cdot v$.

In addition, for brevity, we will often write λv for $\lambda \cdot v$.

EXAMPLE 6.1.2 (The vector space K^n). By definition,
$$K^n = \{(x_1, \ldots, x_n) \ : \ x_i \in K, \ 1 \leq i \leq n\}.$$

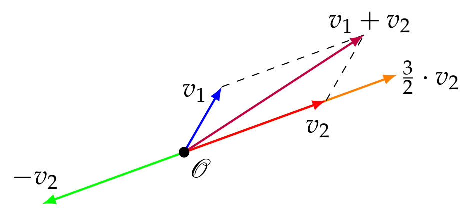

FIGURE 1. Adding and scaling vectors in the plane

The map $+ : K^n \times K^n \to K^n$ is defined by the rule

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in K^n$. The map $\cdot : K \times K^n \to K^n$ is defined by the rule

$$\lambda \cdot (x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$$

for all $\lambda \in K$ and $(x_1, \ldots, x_n) \in K^n$.

Exercise 6.1.3. *Show that $(K^n, +, \cdot)$, defined in the example above, is a K-vector space.*

Exercise 6.1.4 (Cancelation rule). *Let $(V, +, \cdot)$ be a K-vector space. Show that if we have $v_1, v_2, w \in V$, then*

$$v_1 + w = v_2 + w \iff v_1 = v_2.$$

Exercise 6.1.5 (Unique additive identity). *Let $(V, +, \cdot)$ be a K-vector space. Fix an element $\mathscr{O} \in V$ such that for all $v \in V$, we have $v + \mathscr{O} = v$. Show that if $w \in V$ satisfies $v' + w = v'$ for all $v' \in V$, then $w = \mathscr{O}$.*

Exercise 6.1.6 (Unique additive inverse). *Let $(V, +, \cdot)$ be a K-vector space. Let $v \in V$. Fix an element $-v \in V$ such that $v + (-v) = \mathscr{O}$. Suppose that there is $w \in V$ such that $v + w = \mathscr{O}$. Show that $w = -v$.*

Exercise 6.1.7. *Let $(V, +, \cdot)$ be a K-vector space. Show the following properties hold for all $v, v_1, v_2 \in V$ and all $\lambda, \lambda_1, \lambda_2 \in K$.*

(1) $0v = \mathscr{O}$.
(2) $\lambda\mathscr{O} = \mathscr{O}$.
(3) $(-\lambda)v = -(\lambda v) = \lambda(-v)$.
(4) *If $\lambda v = \mathscr{O}$, then either $\lambda = 0$ or $v = \mathscr{O}$.*
(5) *If $\lambda v_1 = \lambda v_2$, then either $\lambda = 0$ or $v_1 = v_2$.*
(6) *If $\lambda_1 v = \lambda_2 v$, then either $\lambda_1 = \lambda_2$ or $v = \mathscr{O}$.*
(7) $-(v_1 + v_2) = (-v_1) + (-v_2)$.
(8) $v + v = 2v$, $v + v + v = 3v$, *and in general* $\sum_{i=1}^{n} v = nv$.

Exercise 6.1.8. *Consider the set of maps from a set S to K. Let us denote this set by* $\mathrm{Map}(S, K)$. *Define addition and multiplication maps*

$$+ : \mathrm{Map}(S, K) \times \mathrm{Map}(S, K) \to \mathrm{Map}(S, K)$$

and

$$\cdot : K \times \mathrm{Map}(S, K) \to \mathrm{Map}(S, K)$$

in the following way. For all $f, g \in \mathrm{Map}(S, K)$, set $f + g$ to be the function defined by $(f + g)(x) = f(x) + g(x)$ for all $x \in S$. For all $\lambda \in K$ and all $f \in \mathrm{Map}(S, K)$, set $\lambda \cdot f$

to be the function defined by $(\lambda \cdot f)(x) = \lambda f(x)$ *for all* $x \in S$. Show that if $S \neq \emptyset$ then $(\mathrm{Map}(S,K),+,\cdot)$ is a K-vector space.

2. Sub-vector spaces

Definition 6.2.9 (sub-K-vector space)**.** *Let* $(V,+,\cdot)$ *be a* K*-vector space. A* ***sub-K-vector space*** *of* $(V,+,\cdot)$ *is a* K*-vector space* $(V',+',\cdot')$ *such that* $V' \subseteq V$ *and such that for all* $v', v_1', v_2' \in V'$ *and all* $\lambda \in K$,

$$v_1' +' v_2' = v_1' + v_2' \quad \text{and} \quad \lambda \cdot' v' = \lambda \cdot v'.$$

We will write $(V',+',\cdot') \subseteq (V,+,\cdot)$.

Definition 6.2.10. *If* $(V,+,\cdot)$ *is a* K*-vector space, and* $V' \subseteq V$ *is a subset, we say that* V' *is* ***closed under*** $+$ ***(resp. closed under*** $\cdot$***)*** *if for all* $v_1', v_2' \in V'$ *(resp. for all* $\lambda \in K$ *and all* $v' \in V'$*) we have* $v_1' + v_2' \in V'$ *(resp.* $\lambda \cdot v' \in V'$*). In this case, we define*

$$+|_{V'} : V' \times V' \to V'$$

(resp. $\cdot|_{V'} : K \times V' \to V'$*) to be the map given by* $v_1' +|_{V'} v_2' = v_1' + v_2'$ *(resp.* $\lambda \cdot|_{V'} v' = \lambda \cdot v'$*), for all* $v_1', v_2' \in V'$ *(resp. for all* $\lambda \in K$ *and all* $v' \in V'$*).*

REMARK 6.2.11. Note that if $(V',+',\cdot')$ is a sub-K-vector space of $(V,+,\cdot)$, then V' is closed under $+$ and $\cdot$.

Exercise 6.2.12. *Show that if a non-empty subset* $V' \subseteq V$ *is closed under* $+$ *and* $\cdot$*, then* $(V', +|_{V'}, \cdot|_{V'})$ *is a sub-*K*-vector space of* $(V,+,\cdot)$.

In other words, in the end, we tend to view a sub-K-vector space

$$(V',+',\cdot') \subseteq (V,+,\cdot)$$

as a subset $V' \subseteq V$ that is closed under $+$ and $\cdot$.

Exercise 6.2.13. *Show that if* $(V',+',\cdot')$ *is a sub-*K*-vector space of a* K*-vector space* $(V,+,\cdot)$*, then the additive identity element* $\mathscr{O}' \in V'$ *is equal to the additive identity element* $\mathscr{O} \in V$.

Exercise 6.2.14. *Recall the* $\mathbb{R}$*-vector space* $(\mathrm{Map}(\mathbb{R},\mathbb{R}),+,\cdot)$ *from Exercise 6.1.8. In this exercise, show that the subsets of* $\mathrm{Map}(\mathbb{R},\mathbb{R})$ *listed below are closed under* $+$ *and* $\cdot$*, and so define sub-*$\mathbb{R}$*-vector spaces of* $(\mathrm{Map}(\mathbb{R},\mathbb{R}),+,\cdot)$.

(1) *The set of all polynomial functions.*
(2) *The set of all polynomial functions of degree less than* n.
(3) *The set of all functions that are continuos on an interval* $(a,b) \subseteq \mathbb{R}$.
(4) *The set of all functions differentiable at a point* $a \in \mathbb{R}$.
(5) *The set of all functions differentiable on an interval* $(a,b) \subseteq \mathbb{R}$.
(6) *The set of all functions with* $f(1) = 0$.
(7) *The set of all solutions to the differential equation* $f'' + af' + bf = 0$ *for some* $a, b \in \mathbb{R}$.

Exercise 6.2.15. *In this exercise, show that the subsets of* $\mathrm{Map}(\mathbb{R},\mathbb{R})$ *listed below are NOT closed under* $+$ *and* $\cdot$*, and so do not define sub-*$\mathbb{R}$*-vector spaces of* $(\mathrm{Map}(\mathbb{R},\mathbb{R}),+,\cdot)$.

(1) *Fix* $a \in \mathbb{R}$ *with* $a \neq 0$*. The set of all functions with* $f(1) = a$.
(2) *The set of all solutions to the differential equation* $f'' + af' + bf = c$ *for some* $a, b, c \in \mathbb{R}$ *with* $c \neq 0$.

3. Linear maps

Definition 6.3.16 (Linear map)**.** *Let $(V, +, \cdot)$ and $(V', +', \cdot')$ be K-vector spaces. A* ***linear map*** *$F : (V, +, \cdot) \to (V', +', \cdot')$ is a map of sets*

$$f : V \to V'$$

such that for all $\lambda \in K$ and $v, v_1, v_2 \in V$,

$$f(v_1 + v_2) = f(v_1) +' f(v_2) \quad \textit{and} \quad f(\lambda \cdot v) = \lambda \cdot' f(v).$$

Note that we will frequently use the same letter for the linear map and the map of sets. The K-vector space $(V, +, \cdot)$ is called the **source** (or domain) of the linear map and the K-vector space $(V', +', \cdot')$ is called the **target** (or codomain) of the linear map. The set $f(V) \subseteq V'$ is called the **image** (or range) of f.

Exercise 6.3.17. *Let $f : (V, +, \cdot) \to (V', +', \cdot')$ be a linear map of K-vector spaces. Show that the image of f is closed under $+', \cdot'$, and so defines a sub-K-vector space of the target $(V', +', \cdot')$.*

Exercise 6.3.18. *Let $f : (V, +, \cdot) \to (V', +', \cdot')$ be a linear map of K-vector spaces. Show that $f(\mathscr{O}) = \mathscr{O}'$.*

Exercise 6.3.19. *Show that the following maps of sets define linear maps of the K-vector spaces.*

(1) *Let $(V, +, \cdot)$ be a K-vector space. Show that the identity map $f : V \to V$, given by $f(v) = v$ for all $v \in V$, is a linear map. This linear map will frequently be denoted by Id_V.*
(2) *Let $(V, +, \cdot)$ and $(V', +', \cdot')$ be K-vector spaces. Show that the zero map $f : V \to V'$, given by $f(v) = \mathscr{O}'$ for all $v \in V$, is a linear map.*
(3) *Let $(V, +, \cdot)$ be a K-vector space and let $\alpha \in K$. Show that the multiplication map $f : V \to V$ given by $f(v) = \alpha \cdot v$ for all $v \in V$ is a linear map. This linear map will frequently be denoted by $\alpha\,\mathrm{Id}_V$.*
(4) *Let $a_{ij} \in K$ for $1 \le i \le m$ and $1 \le j \le n$. Show that the map $f : K^n \to K^m$ given by*

$$f(x_1, \dots, x_n) = \left(\sum_{j=1}^{n} a_{1j} x_j, \dots, \sum_{j=1}^{n} a_{ij} x_j, \dots, \sum_{j=1}^{n} a_{mj} x_j \right)$$

is a linear map.
(5) *Let $(V, +, \cdot)$ be the $\mathbb{R}$-vector space of all differentiable real functions $g : \mathbb{R} \to \mathbb{R}$. Let $(V', +', \cdot')$ be the $\mathbb{R}$-vector space of all real functions $g : \mathbb{R} \to \mathbb{R}$. Show that the map $f : (V, +, \cdot) \to (V', +', \cdot')$ that sends a differentiable function g to its derivative g' is a linear map.*
(6) *Let $(V, +, \cdot)$ be the $\mathbb{R}$-vector space of all continuous real functions $g : \mathbb{R} \to \mathbb{R}$. Show that the map $f : (V, +, \cdot) \to (V, +, \cdot)$ that sends a function $g \in V$ to the function $f(g) \in V$ determined by*

$$f(g)(x) := \int_a^x g(t)dt \quad \textit{for all } x \in \mathbb{R}$$

is a linear map. Make sure to show that $f(g) \in V$ for all $g \in V$.

Definition 6.3.20 (Kernel). *Let $f : (V, +, \cdot) \to (V', +', \cdot')$ be a linear map of K-vector spaces. The* ***kernel of*** *f (or Null space of f), denoted $\ker(f)$ (or $\mathrm{Null}(f)$), is the set*

$$\ker(f) := f^{-1}(\mathscr{O}') = \{v \in V : f(v) = \mathscr{O}'\}.$$

Exercise 6.3.21. *Let $f : (V, +, \cdot) \to (V', +', \cdot')$ be a linear map of K-vector spaces. Show that $\ker(f)$ is a sub-K-vector space of $(V, +, \cdot)$.*

Exercise 6.3.22. *Find the kernel of each of the linear maps listed below (see Problem 6.3.19).*

(1) *The linear map Id_V.*
(2) *The zero map $V \to V'$.*
(3) *The linear map $\alpha\,\mathrm{Id}_V$.*
(4) *Let $a_{ij} \in K$ for $1 \le i \le m$ and $1 \le j \le n$. The linear map $f : K^n \to K^m$ defined by*

$$f(x_1, \ldots, x_n) = \left(\sum_{j=1}^{n} a_{1j}x_j, \ldots, \sum_{j=1}^{n} a_{ij}x_j, \ldots, \sum_{j=1}^{n} a_{mj}x_j \right).$$

(5) *Let $(V, +, \cdot)$ be the $\mathbb{R}$-vector space of all differentiable real functions $g : \mathbb{R} \to \mathbb{R}$. Let $(V', +', \cdot')$ be the $\mathbb{R}$-vector space of all real functions $g : \mathbb{R} \to \mathbb{R}$. The linear map $f : (V, +, \cdot) \to (V', +', \cdot')$ that sends a differentiable function g to its derivative g'.*
(6) *Let $(V, +, \cdot)$ be the $\mathbb{R}$-vector space of all continous real functions $g : \mathbb{R} \to \mathbb{R}$. Let $a \in \mathbb{R}$. The linear map $f : (V, +, \cdot) \to (V, +, \cdot)$ that sends a function $g \in V$ to the function $f(g) \in V$ determined by*

$$f(g)(x) := \int_a^x g(t)dt \quad \text{for all } x \in \mathbb{R}.$$

Exercise 6.3.23. *Show that the composition of linear maps is a linear map.*

Definition 6.3.24 (Isomorphism). *Let $f : (V, +, \cdot) \to (V', +', \cdot')$ be a linear map of K-vector spaces. We say that f is an isomorphism of K-vector spaces if there is a linear map $g : (V', +', \cdot') \to (V, +, \cdot)$ of K-vector spaces such that*

$$g \circ f = \mathrm{Id}_V \quad \text{and} \quad f \circ g = \mathrm{Id}_{V'}.$$

Exercise 6.3.25. *Show that a linear map is an isomorphism if and only if it is bijective.*

4. Bases and dimension

4.1. Linear maps determined by elements of a vector space. The basic example we are interested in is the following. Let V be a K-vector space. We fix

$$\mathbf{v} = (v_1, \ldots, v_n) \in V^n.$$

From this we obtain a map

$$L_{\mathbf{v}} : K^n \to V$$

$$(a_1, \ldots, a_n) \mapsto \sum_{i=1}^{n} a_i v_i.$$

Exercise 6.4.26. *Show that $L_{\mathbf{v}}$ is a linear map.*

4.2. Span, linear independence, and bases. For every permutation $\sigma \in \Sigma_n$, the symmetric group on n-letters, we set

$$\mathbf{v}^\sigma := (v_{\sigma(1)}, \ldots, v_{\sigma(n)}).$$

Definition 6.4.27. *Let V be a K-vector space, and let $v_1, \ldots, v_n \in V$. Set $\mathbf{v} = (v_1, \ldots, v_n)$. We say:*

(1) *The elements $v_1, \ldots, v_n$ **span** V (or generate V) if for every $\sigma \in \Sigma_n$, the linear map $L_{\mathbf{v}^\sigma}$ is surjective.*
(2) *The elements $v_1, \ldots, v_n$ are **linearly independent** if for every $\sigma \in \Sigma_n$, the linear map $L_{\mathbf{v}^\sigma}$ is injective.*
(3) *The elements $v_1, \ldots, v_n$ are a **basis for** V if for every $\sigma \in \Sigma_n$, the linear map $L_{\mathbf{v}^\sigma}$ is an isomorphism.*

Exercise 6.4.28. *Let V be a K-vector space, and let $v_1, \ldots, v_n \in V$. Set $\mathbf{v} = (v_1, \ldots, v_n)$.*

(1) *The elements $v_1, \ldots, v_n$ **span** V (or generate V) if for any $\sigma \in \Sigma_n$, the linear map $L_{\mathbf{v}^\sigma}$ is surjective.*
(2) *The elements $v_1, \ldots, v_n$ are **linearly independent** if for any $\sigma \in \Sigma_n$, the linear map $L_{\mathbf{v}^\sigma}$ is injective.*
(3) *The elements $v_1, \ldots, v_n$ are a **basis for** V if for any $\sigma \in \Sigma_n$, the linear map $L_{\mathbf{v}^\sigma}$ is an isomorphism.*

Exercise 6.4.29. *Let V be a K-vector space, and let $v_1, \ldots, v_n \in V$.*

(1) *The elements $v_1, \ldots, v_n$ **span** V (or generate V) if for any $v \in V$, there exists $(a_1, \ldots, a_n) \in K^n$ such that $\sum_{i=1}^n a_i v_i = v$.*
(2) *The elements $v_1, \ldots, v_n$ are **linearly independent** if whenever $(a_1, \ldots, a_n) \in K^n$ and $\sum_{i=1}^n a_i v_i = 0$, we have $(a_1, \ldots, a_n) = 0$.*
(3) *The elements $v_1, \ldots, v_n$ are a **basis for** V if they span V and are linearly independent.*

4.3. Dimension. We start with the following motivational exercise:

Exercise 6.4.30. *If $K^n \cong K^m$, then $n = m$.*

Definition 6.4.31. *A K-vector space V is said to be of dimension n if there is an isomorphism $V \cong K^n$.*

Exercise 6.4.32. *Show that a K-vector space V has dimension n if and only if it has a basis consisting of n elements.*

5. Direct products of vector spaces

EXAMPLE 6.5.33. Suppose that $(V_1, +_1, \cdot_1)$ and $(V_2, +_2, \cdot_2)$ are K-vector spaces. There is a K-vector space

$$(V_1, +_1, \cdot_1) \times (V_2, +_2, \cdot_2) := (V_1 \times V_2, +, \cdot)$$

where $V_1 \times V_2$ is the product of the sets V_1 and V_2, where

$$+ : (V_1 \times V_2) \times (V_1 \times V_2) \to V_1 \times V_2$$

is defined by

$$(v_1, v_2) + (v_1', v_2') = (v_1 +_1 v_1', v_2 +_2 v_2')$$

and

$$\cdot : K \times (V_1 \times V_2) \to V_1 \times V_2$$

is defined by

$$\lambda \cdot (v_1, v_2) = (\lambda \cdot_1 v_1, \lambda \cdot_2 v_2).$$

Exercise 6.5.34. *Show that the triple* $(V_1, +_1, \cdot_1) \times (V_2, +_2, \cdot_2) := (V_1 \times V_2, +, \cdot)$ *in the example above is a K-vector space.*

Definition 6.5.35 (Direct product)**.** *Suppose that* $(V_1, +_1, \cdot_1)$ *and* $(V_2, +_2, \cdot_2)$ *are K-vector spaces. We define the* direct product *of* $(V_1, +_1, \cdot_1)$ *and* $(V_2, +_2, \cdot_2)$*, written* $(V_1, +_1, \cdot_1) \times (V_2, +_2, \cdot_2)$*, to be the K-vector space* $(V_1 \times V_2, +, \cdot)$ *defined above.*

Exercise 6.5.36. *Let* V_1 *and* V_2 *be K-vector spaces. Show the following:*

(1) *There is an injective linear map* $i_1 : V_1 \to V_1 \times V_2$ *given by* $v_1 \mapsto (v_1, \mathscr{O}_{V_2})$*, and a surjective linear map* $p_1 : V_1 \times V_2 \to V_1$ *given by* $(v_1, v_2) \mapsto v_1$*.*

(2) *There is an injective linear map* $i_2 : V_1 \to V_1 \times V_2$ *given by* $v_2 \mapsto (\mathscr{O}_{V_1}, v_2)$*, and a surjective linear map* $p_2 : V_1 \times V_2 \to V_2$ *given by* $(v_1, v_2) \mapsto v_2$*.*

6. Quotient vector spaces

Suppose that $(V, +, \cdot)$ is a K-vector space, and $W \subseteq V$ is a sub-K-vector space. Define an equivalence relation on V by the rule

$$v_1 \sim v_2 \iff v_1 - v_2 \in W.$$

Exercise 6.6.37. *Show that this defines an equivalence relation on* V*.*

Let V/W be the set of equivalence classes, and let

$$\pi : V \longrightarrow V/W$$

be the quotient map of sets. For any element $\nu \in V/W$, there is an element $v \in V$ such that $\nu = [v]$, where $[v]$ is the equivalence class of v.

Exercise 6.6.38. *Let* V *be a K-vector space and suppose that* $W \subseteq V$ *is a sub-K-vector space.*

(1) *Suppose that* $[v_1], [v_2] \in V/W$*. Show that the rule*

$$[v_1] + [v_2] = [v_1 + v_2]$$

defines a map

$$+ : V/W \times V/W \to V/W.$$

(2) *Suppose that* $\lambda \in K$ *and* $[v] \in V/W$*. Show that the rule*

$$\lambda \cdot [v] = [\lambda \cdot v]$$

defines a map

$$\cdot : K \times V/W \to V/W.$$

(3) *Show that* V/W *is a K-vector space with* $+$ *and* $\cdot$ *defined as above.*

(4) *Show that* $\pi : V \to V/W$ *is a surjective linear map with kernel* W*.*

Definition 6.6.39 (Quotient K-vector space)**.** *Let* V *be a K-vector space and let* $W \subseteq V$ *be a sub-K-vector space. The quotient (K-vector space) of* V *by* W *is the K-vector space* V/W *constructed above.*

Exercise 6.6.40. *Suppose that* $\phi : V \twoheadrightarrow V'$ *is a surjective linear map of K-vector spaces.*

(1) *Show that* $V' \cong V/\ker\phi$*.*

(2) *If* V' *is finite dimensional, show that* $V \cong (\ker\phi) \times V'$*.*

(3) *If* V *and* V' *are finite dimensional, show that* $\dim V = \dim V' + \dim(\ker\phi)$*.*

7. Further exercises

Exercise 6.7.41. *Find an example of a triple $(V, +, \cdot)$ satisfying all of the conditions of the definition of a K-vector space, except for condition (3)(d).*

Exercise 6.7.42. *Suppose that $L : K^n \to K^m$ is a linear map. For $j = 1, \ldots, n$ define $e_j = (0, \ldots, 1, \ldots, 0) \in K^n$ to be the element with all entries 0 except for the j-th place, which is 1. Similarly, for $i = 1, \ldots, m$ define $f_i^\vee : K^m \to K$ to be the linear map defined by $(y_1, \ldots, y_m) \mapsto y_i$. Show that L is the same as the linear map defined in Example 6.3.19(4) with the matrix $A \in \mathrm{M}_{m \times n}(K)$ defined by $A_{ij} = a_{ij} = f_i^\vee(L(e_j))$.*

CHAPTER 7

Reduced row echelon form of a matrix

1. Definitions

The reduced row echelon form of a given matrix is a special matrix obtained from the original matrix by taking linear combinations of the rows. These can be quite useful, for instance in giving a solution to the exercise that dimension is well-defined.

Definition 7.1.1 (Reduced row echelon form)**.** *A matrix is in* reduced row echelon form *if the following hold:*

- *All nonzero rows (rows with at least one nonzero element) are above any rows of all zeroes (i.e., all zero rows, if any, belong at the bottom of the matrix).*
- *The leading coefficient (the first nonzero number from the left, also called the pivot) of a nonzero row is always strictly to the right of the leading coefficient of the row above it.*
- *Every leading coefficient is* 1 *and is the only nonzero entry in its column.*

$$\begin{bmatrix} 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

FIGURE 1. A matrix in reduced row echelon form

Definition 7.1.2 (Elementary row operations)**.** *Let A and B be matrices of the same size. We say that B is obtained from A by an elementary row operation if one of the following hold:*

(1) *B is obtained from A by interchanging two rows;*
(2) *B is obtained from A by multiplying a row of A by a nonzero scalar;*
(3) *B is obtained from A by adding a scalar multiple of one row of A to another.*

We say that B is obtained from A by elementary row operations if there is a finite sequence of matrices $A = A_0, A_1, \ldots, A_n = B$, with A_{i+1} obtained from A_i, $i = 1, \ldots, n-1$, by an elementary row operation.

Exercise 7.1.3. *Given a matrix A show that there is a unique matrix that is in reduced row echelon form that can be obtained from A by elementary row operations. [Hint: use induction on the number of columns of A.]*

Definition 7.1.4. *The matrix obtained from A in the previous exercise is called the reduced row echelon form of A.*

Exercise 7.1.5. *Show that the rows of A are linearly dependent if and only if the reduced row echelon form of A has a zero row.*