MIDTERM I
LINEAR ALGEBRA

MATH 2135

Friday February 16, 2018.

Name	

PRACTICE EXAM

SOLUTIONS

Please answer the all of the questions, and show your work.
You must explain your answers to get credit.
You will be graded on the clarity of your exposition!

1	2	3	4	5	
20	20	20	20	20	total

Date: February 12, 2018.

1. Give the definition of a vector space.

1
20 points

SOLUTION

See the pdf online:

`www.math.colorado.edu/~casa/teaching/18spring/2135/hw/LinAlgHW.pdf`

2
20 points

2. Let V be the $\mathbb{R}$-vector space of sequences of real numbers $\{x_n\}_{n=0}^{\infty}$ with addition and scaling given by:

$$\{x_n\}_{n=0}^{\infty} + \{y_n\}_{n=0}^{\infty} = \{x_n + y_n\}_{n=0}^{\infty}, \quad \lambda \cdot \{x_n\}_{n=0}^{\infty} = \{\lambda x_n\}_{n=0}^{\infty},$$

for all $\{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty} \in V$ and $\lambda \in \mathbb{R}$.

Let $L^1 \subseteq V$ be the subset of sequences $\{x_n\}_{n=0}^{\infty} \in V$ such that $\sum_{n=0}^{\infty} |x_n|$ converges. Let $L^2 \subseteq V$ be the subset of sequences $\{x_n\}_{n=0}^{\infty} \in V$ such that $\sum_{n=0}^{\infty} x_n^2$ converges.

2.(a). Show that L^1 is a subspace of V.

2.(b). Show that L^2 is a subspace of V.

2.(c). Is L^1 contained in L^2?

2.(d). Is L^2 contained in L^1?

SOLUTION

We will repeatedly use the following fact from Calculus II:

> $(*)$ *Given a series $\sum_{n=0}^{\infty} a_n$ with all the a_n being* <u>nonnegative</u> *real numbers, if there exists a real number M such that for all natural numbers N we have $\sum_{n=0}^{N} a_n \leq M$, then $\sum_{n=0}^{\infty} a_n$ is convergent.*

(a) We must show that L^1 is closed under addition and scaling. To show it is closed under addition, let $x = \{x_n\}_{n=0}^{\infty}, y = \{y_n\}_{n=0}^{\infty} \in L^1$. Since $x + y = \{x_n + y_n\}_{n=0}^{\infty}$, we must show that $\sum_{n=0}^{\infty} |x_n + y_n|$ is convergent. To this end, for any natural number N, we have using the triangle inequality (for real numbers) that

$$\sum_{n=0}^{N} |x_n + y_n| \leq \sum_{n=0}^{N} |x_n| + |y_n| = \sum_{n=0}^{N} |x_n| + \sum_{n=0}^{N} |y_n| \leq \sum_{n=0}^{\infty} |x_n| + \sum_{n=0}^{\infty} |y_n|.$$

Thus the series $\sum_{n=0}^{\infty} |x_n + y_n|$ converges, using the fact $(*)$ above.

To show that L^1 is closed under scaling, let $\lambda \in \mathbb{R}$ and $x = \{x_n\}_{n=0}^{\infty} \in L^1$. We must show that $\sum_{n=0}^{\infty} |\lambda x_n|$ is convergent. To this end, for any natural number N we have

$$\sum_{n=0}^{N} |\lambda x_n| = |\lambda| \sum_{n=0}^{N} |x_n| \leq |\lambda| \sum_{n=0}^{\infty} |x_n|;$$

again we conclude using $(*)$.

(b) We must show that L^2 is closed under addition and scaling. To show it is closed under addition, let $x = \{x_n\}_{n=0}^{\infty}, y = \{y_n\}_{n=0}^{\infty} \in L^2$. We must show that $\sum_{n=0}^{\infty} (x_n + y_n)^2$

is convergent. For this we have

$$\sum_{n=0}^{\infty}(x_n+y_n)^2=\sum_{n=0}^{\infty}x_n^2+\sum_{n=0}^{\infty}y_n^2+\sum_{n=0}^{\infty}x_ny_n,$$

assuming the series on the right are all convergent. Thus it suffices to show that $\sum_{n=0}^{\infty}x_ny_n$ is convergent, and therefore to show that the series $\sum_{n=0}^{\infty}|x_ny_n|$ is convergent.

To this end, first observe that it is clear that $\{|x_n|\},\{|y_n|\}\in L^2$. Now, using the fact that the standard dot product on $\mathbb{R}^{N+1}$ is an inner product, we have from Cauchy–Schwartz that

$$\begin{aligned}\sum_{n=0}^{N}|x_ny_n|=|\sum_{n=0}^{N}|x_n||y_n||&\le\left(\sum_{n=0}^{N}|x_n|^2\right)^{1/2}\left(\sum_{n=0}^{N}|y_n|^2\right)^{1/2}\\&\le\left(\sum_{n=0}^{\infty}|x_n|^2\right)^{1/2}\left(\sum_{n=0}^{\infty}|y_n|^2\right)^{1/2}\end{aligned}$$

Again we conclude using $(*)$.

To show that L^2 is closed under scaling, let $\lambda\in\mathbb{R}$ and $x=\{x_n\}_{n=0}^{\infty}\in L^2$. We must show that $\sum_{n=0}^{\infty}(\lambda x_n)^2$ is convergent. For this we have

$$\sum_{n=0}^{N}(\lambda x_n)^2=\lambda^2\sum_{n=0}^{N}x_n^2\le\lambda^2\sum_{n=0}^{\infty}x_n^2;$$

again we conclude using $(*)$.

(c) $\boxed{\textit{Yes.}}$ The subspace L^1 is contained in the subspace L^2. Indeed, suppose that $\{x_n\}\in L_1$. Then we have $\sum_{n=0}^{\infty}|x_n|$ converges. Now there is some natural number N_0 such that $0\le|x_n|<1$ for all $n\ge N_0$. Therefore, for all $n\ge N_0$, we have $0\le x_n^2\le|x_n|$, so that consequently, for all $N\ge N_0$

$$\sum_{n=0}^{N}x_n^2=\sum_{n=0}^{N_0-1}x_n^2+\sum_{n=N_0}^{N}x_n^2\le\sum_{n=0}^{N_0-1}x_n^2+\sum_{n=N_0}^{N}|x_n|\le\sum_{n=0}^{N_0-1}x_n^2+\sum_{n=N_0}^{\infty}|x_n|.$$

Thus by $(*)$ the series $\sum_{n=0}^{\infty}x_n^2$ converges, so that $\{x_n\}\in L^2$.

(d) $\boxed{\textit{No.}}$ The subspace L^2 is not contained in the subspace L^1. For instance taking $x_n=\frac{1}{n}$ for all $n\ge 1$ (and x_0 arbitrary), we have $\{x_n\}\in L^2$, but $\{x_n\}\notin L^1$ (one can use the integral test, for instance, for both of these).

3
20 points

3. Let L^2 be the $\mathbb{R}$-vector space consisting of sequences of real numbers $\{x_n\}_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty} x_n^2$ converges. We have seen there is an inner product on L^2 defined by setting

$$(x, y) = \sum_{n=0}^{\infty} x_n y_n$$

for $x = \{x_n\}_{n=0}^{\infty}, y = \{y_n\}_{n=0}^{\infty} \in L^2$.

Now consider the sequences $\{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty}$ defined by:

$$x_n = \frac{1}{2^n},$$
$$y_n = \begin{cases} 1 & n = 0, \\ 0 & n > 0. \end{cases}$$

3.(a). Perform the Graeme–Schmidt process to obtain an orthonomal basis for the vector space spanned by $\{x_n\}$ and $\{y_n\}$.

3.(b). Consider the sequence $\{z_n\}_{n=0}^{\infty}$ defined by $z_n = 1/n!$. Find the orthogonal projection of $\{z_n\}$ onto the vector space spanned by $\{x_n\}$ and $\{y_n\}$.

SOLUTION

We start with a few useful computations, just to have them all in one place:

$$(x, x) = \sum_{n=0}^{\infty} \left(\frac{1}{2^n}\right)^2 = \sum_{n=0}^{\infty} \left(\frac{1}{2^2}\right)^n = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3},$$
$$(y, x) = 1 \cdot 1 + 0 \cdot \frac{1}{2} + \cdots = 1,$$
$$(y, y) = 1 \cdot 1 + 0 \cdot 0 + \cdots = 1,$$
$$(z, x) = \sum_{n=0}^{\infty} \frac{1}{n!}(1/2)^n = e^{1/2},$$
$$(z, y) = 1 \cdot 1 + 1 \cdot 0 + \frac{1}{2} \cdot 0 + \cdots = 1.$$

(a) The solution to the problem is:

$$\boxed{\begin{aligned} x'' &= \frac{\sqrt{3}}{2}x \\ y'' &= 2y - \frac{3}{2}x \end{aligned}}$$

The proof is as follows. The Graeme–Schmidt process starts by constructing an orthogonal basis:

$$\begin{aligned}
x' &= x. \\
y' &= y - \frac{(y,x)}{(x,x)}x \\
&= y - \frac{1}{4/3}x \\
y' &= y - \frac{3}{4}x.
\end{aligned}$$

I.e., the vectors x', y' form an orthogonal basis for the subspace spanned by x and y.

To obtain an orthonormal basis, we should compute:

$$\begin{aligned}
(y', y') = (y - \frac{3}{4}x, y - \frac{3}{4}x) &= (y,y) + \frac{9}{16}(x,x) - 2\frac{3}{4}(x,y) \\
&= 1 + \frac{9}{16}\cdot\frac{4}{3} - \frac{3}{2}\cdot 1 = \frac{1}{4}.
\end{aligned}$$

Thus

$$\begin{aligned}
x'' &= \frac{x'}{(x',x')^{1/2}} = \frac{x}{(x,x)^{1/2}} = \frac{\sqrt{3}}{2}x \\
y'' &= \frac{y'}{(y',y')^{1/2}} = 2y' = 2y - \frac{3}{2}x
\end{aligned}$$

form an orthonormal basis for the subspace spanned by x and y. Specifically, we have

$$\begin{aligned}
x''_n &= \frac{\sqrt{3}}{2}\frac{1}{2^n} = \frac{\sqrt{3}}{2^{n+1}} \\
y''_n &= \begin{cases} 2 - \frac{3}{2}\cdot\frac{1}{2^n} = \frac{1}{2} & n = 0 \\ -\frac{3}{2}\cdot\frac{1}{2^n} = -\frac{3}{2^{n+1}} & n > 0 \end{cases}
\end{aligned}$$

Remark 0.1. We can do a quick double check that x'' and y'' are the correct solution. We clearly have that x'' and y'' are in the span of x and y, and that x'' is in the span of x. We have $(x'', x'') = (\frac{\sqrt{3}}{2}x, \frac{\sqrt{3}}{2}x) = \frac{3}{4}(x,x) = 1$, $(y'', y'') = (2y - \frac{3}{2}x, 2y - \frac{3}{2}x) = 4(y,y) - 2\frac{3}{2}2(x,y) + \frac{9}{4}(x,x) = 4 - 6 + \frac{9}{4}\frac{4}{3} = 1$, $(x'', y'') = (\frac{\sqrt{3}}{2}x, 2y - \frac{3}{2}x) = \sqrt{3}(x,y) - \frac{\sqrt{3}}{2}\frac{3}{2}(x,x) = \sqrt{3}(x,y) - \frac{3\sqrt{3}}{4}(x,x) = \sqrt{3}\cdot 1 - \frac{3\sqrt{3}}{4}\cdot\frac{4}{3} = 0$.

(b) The solution to the problem is:

$$\boxed{w = (-3 + 3e^{1/2})x + (4 - 3e^{1/2})y} \tag{0.1}$$

The proof is as follows. Having computed the orthonormal basis x'', y'' in the previous part of the problem, the orthogonal projection $w = \{w_n\}$ of $z = \{z_n\}$ is given by

$$w = (z, x'')x'' + (z, y'')y''.$$

We compute:

$$(z, x'') = (z, \frac{\sqrt{3}}{2}x) = \frac{\sqrt{3}}{2}(z, x) = \frac{\sqrt{3}}{2}e^{1/2}.$$
$$(z, y'') = (z, 2y - \frac{3}{2}x) = 2(z, y) - \frac{3}{2}(z, x) = 2 - \frac{3}{2}e^{1/2}.$$

Therefore finally, we have

$$\begin{aligned}
w &= (z, x'')x'' + (z, y'')y'' \\
&= \frac{\sqrt{3}}{2}e^{1/2}(\frac{\sqrt{3}}{2}x) + (2 - \frac{3}{2}e^{1/2})(2y - \frac{3}{2}x)) \\
&= [\frac{3}{4}e^{1/2} + (-3 + \frac{9}{4}e^{1/2})]x + (4 - 3e^{1/2})y \\
&= (-3 + 3e^{1/2})x + (4 - 3e^{1/2})y
\end{aligned}$$

In other words, the solution to the problem is that the orthogonal projection of z is given by (0.1).

Specifically, we have

$$w_n = \begin{cases} 1 & n = 0 \\ (-3 + 3e^{1/2})\frac{1}{2^n} & n > 0 \end{cases}$$

Remark 0.2. We can check that we have not made an arithmetic mistake in (0.1) as follows. We clearly have w in the span of x and y. To check that w is the orthogonal projection, we can check that $w^{\perp} := z - w$ is in the orthogonal complement of the span of x and y. In other words, we can check that $(z - w, x) = (z - w, y) = 0$.

$$\begin{aligned}
(z - w, x) &= (z, x) - (w, x) \\
&= e^{1/2} - ((-3 + 3e^{1/2})x + (4 - 3e^{1/2})y, x) \\
&= e^{1/2} - [(-3 + 3e^{1/2})(x, x) + (4 - 3e^{1/2})(y, x)] \\
&= e^{1/2} - [(-3 + 3e^{1/2})\frac{4}{3} + (4 - 3e^{1/2})] \\
&= 0. \\
(z - w, y) &= (z, y) - (w, y) \\
&= 1 - [((-3 + 3e^{1/2})x + (4 - 3e^{1/2})y, y)] \\
&= 1 - [(-3 + 3e^{1/2})(x, y) + (4 - 3e^{1/2})(y, y)] \\
&= 1 - [(-3 + 3e^{1/2}) + (4 - 3e^{1/2})] \\
&= 0.
\end{aligned}$$

4
20 points

4. Let $(V, (-,-))$ be a real Euclidean space. Let W be the vector space of linear maps $L : V \to \mathbb{R}$.

4.(a). For each $v \in V$, show that the map of sets

$$(-,v) : V \longrightarrow \mathbb{R}$$
$$(-,v)(v') = (v', v)$$

is a linear map.

4.(b). Show that the map of sets

$$\phi : V \longrightarrow W$$
$$v \mapsto (-,v)$$

is a linear map.

SOLUTION

(a) Let $v \in V$. Then for all $v_1, v_2 \in V$, we have

$$(-,v)(v_1 + v_2) := (v_1 + v_2, v) = (v_1, v) + (v_2, v) =: (-,v)(v_1) + (-,v)(v_2).$$

The second equality above uses the bilinearity of the inner product.

Similarly, for all $\lambda \in \mathbb{R}$, we have

$$(-,v)(\lambda \cdot v_1) := (\lambda \cdot v_1, v) = \lambda \cdot (v_1, v) =: \lambda \cdot (-,v)(v_1).$$

The second equality above uses the bilinearity of the inner product.

(b) For all $v, v_1, v_2 \in V$, and $\lambda \in \mathbb{R}$, we must show that

$$\phi(v_1 + v_2) = \phi(v_1) + \phi(v_2), \quad \phi(\lambda v) = \lambda \phi(v).$$

These are supposed to be equalities of maps $V \to \mathbb{R}$, and it suffices to check that the maps agree for all $x \in V$. First we have

$$\begin{aligned}\phi(v_1 + v_2)(x) &:= (-, v_1 + v_2)(x) := (x, v_1 + v_2) = (x, v_1) + (x, v_2) \\ &=: (-, v_1)(x) + (-, v_2)(x) =: (\phi(v_1) + \phi(v_2))(x).\end{aligned}$$

The third equality above uses the bilinearity of the inner product. Thus $\phi(v_1 + v_2) = \phi(v_1) + \phi(v_2)$.

Similarly, we have

$$\phi(\lambda \cdot v)(x) := (-, \lambda \cdot v)(x) := (x, \lambda \cdot v) = \lambda \cdot (x, v) =: \lambda \cdot (-,v)(x) =: \lambda \cdot \phi(x);$$

the third equality above uses the bilinearity of the inner product.

5
20 points

5. True or False.

Give a *brief* explanation for each answer. This *should be at most a sentence or two* with the main idea, or the main theorem you use, or the example, or the counter example, etc.

5.(a). Every complex vector space is isomorphic to $\mathbb{C}^n$ for some nonnegative integer n.

T F | FALSE. For instance, the complex vector space of polynomials with complex coefficients $\mathbb{C}[x]$ is not finite dimensional (for each natural number n it contains the $(n+1)$-dimensional vector space $\mathbb{C}[x]_n$ of polynomials of dimension at most n).

5.(b). Let V be a vector space, and let $V' \subseteq V$ be a subspace. Then the additive identity of V is equal to the additive identity of V'.

T F | TRUE. This was a homework exercise. We showed for instance that for any $v \in V$, we have $0 \cdot v = \mathscr{O}$. We can then see that $\mathscr{O}' = 0 \cdot' \mathscr{O}' = 0 \cdot \mathscr{O}' = \mathscr{O}$.

5.(c). In the vector space $C^0(\mathbb{R})$ of continuous maps $f : \mathbb{R} \to \mathbb{R}$, the elements $1, \cos t, \sin t$ are linearly independent.

T F | TRUE. If $a + b\cos t + c\sin t = 0$, then taking the derivative, we have $c\cos t = b\sin t$, which is only possibly if $b = c = 0$, since otherwise the values of t for which $c\cos t$ and $b\sin t$ are zero, are different. Then $a = 0$ and we are done.

5.(d). A linear map is surjective if and only if it has trivial kernel.

T F | FALSE. Take for instance $\mathbb{R}^0 \hookrightarrow \mathbb{R}$.

5.(e). Let $(V, (-,-))$ be a Euclidean vector space, and let W be a subspace. Then we have $W \cap W^\perp = \{\mathscr{O}\}$, where $\mathscr{O}$ is the additive identity of V.

T F | TRUE. We saw this in class: if $w \in W \cap W^\perp$, then $0 = (w, w)$, which, from the definition of an inner product, is only possibble if $w = \mathscr{O}$.

5.(f). Let $(V, (-,-))$ be a Euclidean vector space, and let W be a finite dimensional subspace with orthonormal basis $w_1, \ldots, w_n$. Then for any $v \in V$ we have $v - \sum_{i=1}^n (v, w_i) w_i$ is in W.

T F | FALSE. This is true only if $v \in W$, since $v - \sum_{i=1}^n (v, w_i) w_i \in W \iff v = (v - \sum_{i=1}^n (v, w_i) w_i) + \sum_{i=1}^n (v, w_i) w_i \in W$.

5.(g). If $v_1, \ldots, v_n$ are nonzero, mutually orthogonal elements of a Euclidean space, then they are linearly indepenent.

T F | TRUE. Suppose that $\sum_{i=1}^n \alpha_i v_i = 0$. Then for each v_j, $j = 1, \ldots, n$, we have that $0 = (\sum_{i=1}^n \alpha_i v_i, v_j) = \alpha_j \cdot (v_j, v_j)$, which, since $v_j \neq \mathscr{O}$, is only possibly if $\alpha_j = 0$.

5.(h). If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a surjective linear map, then $n \geq m$.

T F | TRUE. The Rank-Nullity Theorem implies that $n = m + \dim\ker f \geq m$.

5.(i). A linear map $f : \mathbb{R}^n \to V$ is injective if and only if $\dim \operatorname{Im}(f) = n$.

T F | TRUE. This was a theorem we proved in class. The argument was that we have f injective $\iff \ker(f) = 0 \iff \dim \operatorname{Im}(f) = n$. The first equivalence was a theorem we proved in class, and the second equivalence follows from the Rank–Nullity theorem.

5.(j). Elements $v_1, \ldots, v_n$ of a vector space V form a basis of V if and only if they span V.

T F | FALSE: The definition requires they also be linearly independent. For instance $(1,0), (0,1), (1,1)$ span $\mathbb{R}^2$, but are not linearly independent, so do not form a basis.