MIDTERM II
LINEAR ALGEBRA

MATH 2135

Friday March 23, 2018.

Name | ______________________________

PRACTICE EXAM

SOLUTIONS

Please answer the all of the questions, and show your work.
You must explain your answers to get credit.
You will be graded on the clarity of your exposition!

1	2	3	4	5	6	
20	20	20	20	20	20	total

Date: March 19, 2018.

1
20 points

1. Let V be an n-dimensional vector space over $K \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$, and let $v_1, \ldots, v_n$ be a basis for V. *Give the definition of a determinant function for the vector space V with respect to the basis $v_1, \ldots, v_n$.*

SOLUTION

A determinant function d for the K-vector space V with respect to the basis $v_1, \ldots, v_n$ is a map

$$d : \underbrace{V \times \cdots \times V}_{n} \to K$$

satisfying:

(1) d is multi-linear; i.e., for any $i = 1, \ldots, n$, given $x_1, \ldots, x_n, y_i \in V$, and $\alpha, \beta \in K$, then we have

$$d(x_1, \ldots, x_{i-1}, \alpha x_i + \beta y_i, x_{i+1}, \ldots, x_n)$$
$$= \alpha d(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) + \beta d(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)$$

(2) d is alternating; i.e., for any $x_1, \ldots, x_n \in V$, if $x_i = x_j$ for $i \neq j$, then

$$d(x_1, \ldots, x_n) = 0.$$

(3) $d(v_1, \ldots, v_n) = 1$.

2
20 points

2. Let V be the real vector space spanned by $1, \cos t, \sin t$ in the real vector space $\mathrm{Diff}(\mathbb{R}, \mathbb{R})$ of differentiable real valued functions.

2.(a). Let $T : V \to V$ be the linear map defined by differentiation, i.e., $T(f) = f'$. *Give the matrix form of T with respect to the basis* $1, \cos t, \sin t$.

2.(b). *Find two bases v_1, v_2, v_3 and w_1, w_2, w_3 for V so that with respect to these bases, the matrix form of T is diagonal.*

More precisely, find two bases v_1, v_2, v_3 and w_1, w_2, w_3 for V so that if the first basis defines an isomorphism $\phi : \mathbb{R}^3 \to V$ and the second defines an isomorphism $\psi : \mathbb{R}^3 \to V$, then the matrix associated to the composition

$$\mathbb{R}^3 \xrightarrow{\phi} V \xrightarrow{T} V \xrightarrow{\psi^{-1}} \mathbb{R}^3$$

is diagonal.

SOLUTION

(a) Associated to the given basis, we obtain an isomorphism $\phi : \mathbb{R}^3 \to V$ given by $\phi(e_1) = 1$, $\phi(e_2) = \cos t$ and $\phi(e_3) = \sin t$. This gives us a linear map

$$L : \mathbb{R}^3 \xrightarrow{\phi} V \xrightarrow{T} V \xrightarrow{\phi^{-1}} \mathbb{R}^3.$$

We are asked to given the matrix form of L. Since we have

$$\begin{aligned} T(\phi(e_1)) &= T(1) = 0 + 0\cos t + 0\sin t = 0 \\ T(\phi(e_2)) &= T(\cos t) = 0 + 0\cos t - \sin t = \phi(-e_3) \\ T(\phi(e_3)) &= T(\sin t) = 0 + \cos t + 0\sin t = \phi(e_2), \end{aligned}$$

it follows that

$$\begin{aligned} L(e_1) &= 0e_1 + 0e_2 + 0e_3 \\ L(e_2) &= 0e_1 + 0e_2 - e_3 \\ L(e_3) &= 0e_1 + e_2 + 0e_3. \end{aligned}$$

Thus, taking the rows above and entering them as columns, the matrix form of L is

$$\boxed{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}}$$

(b) In the computation for (a), we saw that 1 was a basis for the kernel of T and $T(\cos t), T(\sin t)$ were a basis for the image of T. Moreover, $1, T(\cos t), T(\sin t)$ span V.

Thus we may take as our bases:

$$\begin{array}{ll} v_1 = \; 1 & w_1 = 1 \\ v_2 = \; \cos t & w_2 = T(v_2) = -\sin t \\ v_3 = \; \sin t & w_3 = T(v_3) = \cos t \end{array}$$

Indeed, with respect to these bases, we have $\phi : \mathbb{R}^3 \to V$ given by $\phi(e_1) = 1$, $\phi(e_2) = \cos t$ and $\phi(e_3) = \sin t$, and $\psi : \mathbb{R}^3 \to V$ given by $\psi(e_1) = 1$, $\psi(e_2) = -\sin t$ and $\psi(e_3) = \cos t$. The claim is that the matrix form of the linear map

$$L : \mathbb{R}^3 \xrightarrow{\phi} V \xrightarrow{T} V \xrightarrow{\psi^{-1}} \mathbb{R}^3$$

is diagonal. Since we have

$$\begin{aligned} T(\phi(e_1)) &= T(1) = 0 + 0\cos t + 0\sin t = 0 \\ T(\phi(e_2)) &= T(\cos t) = 0 + 0\cos t - \sin t = \psi(e_2) \\ T(\phi(e_3)) &= T(\sin t) = 0 + \cos t + 0\sin t = \psi(e_3). \end{aligned}$$

it follows that

$$\begin{aligned} L(e_1) &= 0e_1 + 0e_2 + 0e_3 \\ L(e_2) &= 0e_1 + e_2 + 0e_3 \\ L(e_3) &= 0e_1 + 0e_2 + e_3. \end{aligned}$$

Thus the matrix form of L is:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is diagonal, as claimed.

3
20 points

3. *Find the reduced row echelon form of the following matrix:*

$$A = \begin{pmatrix} 1 & -3 & 0 & -1 & 4 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 3 & -9 & 0 & -3 & 2 & 4 \\ 1 & -3 & 1 & -2 & 4 & -1 \end{pmatrix}$$

SOLUTION

The RREF of the matrix A is

$$\boxed{\text{RREF}(A) = \begin{pmatrix} 1 & -3 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}$$

Indeed we have

$$\begin{pmatrix} 1 & -3 & 0 & -1 & 4 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 3 & -9 & 0 & -3 & 2 & 4 \\ 1 & -3 & 1 & -2 & 4 & -1 \end{pmatrix}$$

$$\begin{matrix} R_3' = -3R_1 + R_3 \\ R_4' = -R_1 + R_4 \end{matrix} \quad \begin{pmatrix} 1 & -3 & 0 & -1 & 4 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -10 & 10 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{pmatrix}$$

$$\begin{matrix} R_3' = -\frac{1}{10}R_3 \\ R_4' = -R_2 + R_4 \end{matrix} \quad \begin{pmatrix} 1 & -3 & 0 & -1 & 4 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R_1' = R_1 - 4R_3 \quad \begin{pmatrix} 1 & -3 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

4. Let A be the matrix in the previous problem.

4
20 points

4.(a). Let $A^T : \mathbb{R}^4 \to \mathbb{R}^6$ be the linear map associated to the transpose of A. *Find a basis for the image of A^T.*

4.(b). Let $A : \mathbb{R}^6 \to \mathbb{R}^4$ be the linear map associated to A. *Find a basis for the kernel of A.*

4.(c). *Find all real solutions to the system of linear equations:*

$$\begin{array}{rcrcrcrcrcr} x_1 & - & 3x_2 & & & - & x_4 & + & 4x_5 & = & -2 \\ & & & & x_3 & - & x_4 & & & = & 1 \\ 3x_1 & - & 9x_2 & & & - & 3x_4 & + & 2x_5 & = & 4 \\ x_1 & - & 3x_2 & + & x_3 & - & 2x_4 & + & 4x_5 & = & -1 \end{array}$$

SOLUTION

(a) The image is spanned by the columns of A^T, which are the rows of A. We found that

$$\mathrm{RREF}(A) = \begin{pmatrix} 1 & -3 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The nonzero rows of this matrix form a basis for the image of A^T. In other words,

$$\boxed{(1,-3,0,-1,0,2),\ (0,0,1,-1,0,1),\ (0,0,0,0,1,-1)}$$

form a basis for the image of A^T.

(b) We saw that

$$\mathrm{RREF}(A) = \begin{pmatrix} 1 & -3 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Adding rows, we obtain the matrix

$$\begin{pmatrix} 1 & -3 & 0 & -1 & 0 & 2 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

The columns of the matrix with the green -1s form a basis for the kernel. In other words,

$$\boxed{(-3,-1,0,0,0,0),\ (-1,0,-1,-1,0,0),\ (2,0,1,0,-1,-1)}$$

is a basis for the kernel of A.

(c) The system of linear equations is

$$\begin{pmatrix} 1 & -3 & 0 & -1 & 4 \\ 0 & 0 & 1 & -1 & 0 \\ 3 & -9 & 0 & -3 & 2 \\ 1 & -3 & 1 & -2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 4 \\ -1 \end{pmatrix}$$

The associated augmented matrix is the matrix A:

$$\left(\begin{array}{ccccc|c} 1 & -3 & 0 & -1 & 4 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 3 & -9 & 0 & -3 & 2 & 4 \\ 1 & -3 & 1 & -2 & 4 & -1 \end{array}\right)$$

Therefore, the RREF is

$$\left(\begin{array}{ccccc|c} 1 & -3 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Adding rows, we obtain the matrix

$$\left(\begin{array}{ccccc|c} 1 & -3 & 0 & -1 & 0 & 2 \\ \color{red}{0} & \color{green}{-1} & \color{red}{0} & \color{red}{0} & \color{red}{0} & \color{red}{0} \\ 0 & 0 & 1 & -1 & 0 & 1 \\ \color{red}{0} & \color{red}{0} & \color{red}{0} & \color{green}{-1} & \color{red}{0} & \color{red}{0} \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right)$$

Thus the solutions to the system of equations are:

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \in \left\{ t_1 \begin{pmatrix} -3 \\ \color{green}{-1} \\ 0 \\ \color{red}{0} \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -1 \\ \color{red}{0} \\ -1 \\ \color{green}{-1} \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ \color{red}{0} \\ 1 \\ \color{red}{0} \\ -1 \end{pmatrix} : t_1, t_2 \in \mathbb{R} \right\}}$$

5. *Find the determinant of the following matrix:*

5
20 points

$$B = \begin{pmatrix} 1 & -3 & 2 & 0 & -1 \\ 0 & 3 & 1 & -1 & 1 \\ 2 & -6 & 2 & -3 & 2 \\ 0 & 3 & 3 & -2 & 2 \\ 0 & -3 & 1 & 1 & 1 \end{pmatrix}$$

SOLUTION

We have

$$\boxed{\det(B) = 54.}$$

Indeed,

$$\begin{vmatrix} 1 & -3 & 2 & 0 & -1 \\ 0 & 3 & 1 & -1 & 1 \\ 2 & -6 & 2 & -3 & 2 \\ 0 & 3 & 3 & -2 & 2 \\ 0 & -3 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 2 & 0 & -1 \\ 0 & 3 & 1 & -1 & 1 \\ 0 & 0 & -2 & -3 & 4 \\ 0 & 3 & 3 & -2 & 2 \\ 0 & -3 & 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -3 & 2 & 0 & -1 \\ 0 & 3 & 1 & -1 & 1 \\ 0 & 0 & -2 & -3 & 4 \\ 0 & 0 & 2 & -1 & 1 \\ 0 & 0 & 2 & 0 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -3 & 2 & 0 & -1 \\ 0 & 3 & 1 & -1 & 1 \\ 0 & 0 & -2 & -3 & 4 \\ 0 & 0 & 0 & -4 & 5 \\ 0 & 0 & 0 & -3 & 6 \end{vmatrix}$$

$$= (1) \cdot (3) \cdot (-2) \cdot [(-4)(6) - (5)(-3)]$$

$$= (1) \cdot (3) \cdot (-2) \cdot (-9) = 54.$$

6
20 points

6. *For $K \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$, show:*

A matrix $A \in \mathrm{M}_{n\times n}(K)$ has rank $n-1 \iff \det A = 0$ and there is some $(n-1)\times(n-1)$ minor A_{ij} of A with $\det A_{ij} \neq 0$.

SOLUTION

($\Longleftarrow$) Suppose that $\det A = 0$ and $\det A_{ij} \neq 0$ for some $(n-1)\times(n-1)$ minor A_{ij} of A obtained by removing the i-th column and j-th row from A. Since $\det A = 0$, we know that $\mathrm{rk}(A) \leq n-1$. To show that $\mathrm{rk}(A) \geq n-1$, consider the matrix A_i obtained from A by removing the i-th row, namely,

$$A_i = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

This matrix has linearly independent rows. Indeed, if there were a relation among the rows of A_i, then there would be a relation among the rows of the minor A_{ij} (obtained from A_i by removing the j-th column). But then $\det A_{ij} = 0$, contradicting our assumption.

Now, since the rows of A_i are rows of A, we see that A has $n-1$ rows that are linearly independent, establishing that $\mathrm{rk}(A) \geq n-1$, and therefore that $\mathrm{rk}(A) = n-1$.

($\Longrightarrow$) Suppose that $\mathrm{rk}(A) = n-1$. The first observation is that $\det A = 0$. We now need to show that there is some $(n-1)\times(n-1)$ minor A_{ij} of A such that $\det A_{ij} \neq 0$.

We start by observing that since $\mathrm{rk}(A) = n-1$, there are $n-1$ rows of A that are linearly independent. Suppose that these $n-1$ rows are all of the rows of A with the exception of row i, so that the matrix

$$A_i = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

has linearly independent rows. Let $(a_1, \ldots, a_n) \in K^n$ be any vector not in the span of the rows of A_i, and consider the $n \times n$ matrix $\hat{A}$ obtained from A_i by adding the vector

$(a_1, \dots, a_n)$ as the i-th row:

$$\hat{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,n} \\ a_1 & \cdots & a_n \\ a_{i+1,1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

Now since the rows of $\hat{A}$ are linearly independent, we have $\det \hat{A} \neq 0$. Expanding the determinant on the i-th row of $\hat{A}$,

$$\det \hat{A} = \sum_{j=1}^{n} (-1)^{i+j} a_j \det \hat{A}_{ij},$$

we see that there must be some minor $\hat{A}_{ij}$ of $\hat{A}$ with $\det \hat{A}_{ij} \neq 0$. But for all $j = 1, \dots, n$, we have $\hat{A}_{ij} = A_{ij}$, and so we are done.