Math 2300-007: Quiz 9

Name: Solutions Score: ________

1. (5 points) Find the 5th degree Taylor Polynomial for $f(x) = \sin(x)$ centered at $a = 0$.

n	$f^{(n)}(x)$	$f^{(n)}(a) = f^{(n)}(0)$
0	$\sin(x)$	0
1	$\cos(x)$	1
2	$-\sin(x)$	0
3	$-\cos(x)$	-1
4	$\sin(x)$	0
5	$\cos(x)$	1

Consequently, $T_5(x)$, the fifth-degree Taylor Polynomial for $f(x) = \sin(x)$, centered at $a = 0$ is given by

$$\begin{aligned}
&T_5(x) \\
&= f(0) + \frac{f'(0)}{1!}(x-0)^1 + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{(4)}(0)}{4!}(x-0)^4 + \frac{f^{(5)}(0)}{5!}(x-0)^5 \\
&= 0 + \frac{1}{1}x + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 \\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!}.
\end{aligned}$$

2. (5 points) What is the interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(-3)^n}{\sqrt{n}}(5x-1)^n$?

First, we apply the ratio test:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-3)^{n+1}(5x-1)^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{(-3)^n(5x-1)^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{3(5x-1)}{1}\cdot\frac{\sqrt{n}}{\sqrt{n+1}}\right| \\ &= |3(5x-1)|\cdot\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}} \\ &= |3(5x-1)|\cdot 1.\end{aligned}$$

In order for the ratio test to imply that the series converges, we need

$$\begin{gathered}-1 < 3(5x-1) < 1 \\ \frac{-1}{3} < 5x-1 < \frac{1}{3} \\ \frac{2}{3} < 5x < \frac{4}{3} \\ \frac{2}{15} < x < \frac{4}{15}.\end{gathered}$$

This shows that the series converges for x in $\left(\frac{2}{15}, \frac{4}{15}\right)$ and diverges for $x < \frac{2}{15}$ and for $x > \frac{4}{15}$. It remains to check what happens at the endpoints:

$x = \frac{2}{15}$: We have

$$\sum_{n=1}^{\infty}\frac{(-3)^n}{\sqrt{n}}(5\cdot\frac{2}{15}-1)^n = \sum_{n=1}^{\infty}\frac{(-3)^n}{\sqrt{n}}\left(-\frac{1}{3}\right)^n = \sum_{n=1}^{\infty}\frac{(1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}},$$

which diverges by the p-test ($p = 1/2 \le 1$).

$x = \frac{4}{15}$: We have

$$\sum_{n=1}^{\infty}\frac{(-3)^n}{\sqrt{n}}(5\cdot\frac{4}{15}-1)^n = \sum_{n=1}^{\infty}\frac{(-3)^n}{\sqrt{n}}\left(\frac{1}{3}\right)^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}},$$

which converges by the Alternating Series Test.

Consequently, the interval of convergence for the series $\sum_{n=1}^{\infty}\frac{(-3)^n}{\sqrt{n}}(5x-1)^n$ is

$$\left(\frac{2}{15}, \frac{4}{15}\right].$$