Math 2300-007: Quiz 3

Name: Solutions : 2/8/18 Score: ________

Collaborators:

Directions: This take-home quiz will be due at the beginning of class on Tuesday, February 6. You may use your notes, textbook, and colleagues from our class as resources, but your final write-up should be in your own words. If you work with collaborators from our class, please include their names on this quiz.

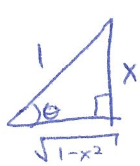
1. Integrate $\int \frac{4x}{(1-x^2)^2}dx$ using the following three techniques:

   (a) (1 point) u/du-substitution;

$$\int \frac{4x}{(1-x^2)^2}\,dx = -\frac{1}{2}\int \frac{4}{u^2}\,du = -2\int u^{-2}\,du = 2u^{-1}+C$$

$$\left\{\begin{array}{l} u = 1-x^2 \\ du = -2x\,dx \\ -\frac{1}{2}du = x\,dx \end{array}\right\} \qquad = \frac{2}{u}+C = \frac{2}{1-x^2}+C$$

   (b) (2 points) trigonometric substitution;



$\sin\theta = x$
$\cos\theta\, d\theta = dx$

$\cos\theta = \sqrt{1-x^2}$
$\cos^4\theta = (1-x^2)^2$

$$\int \frac{4x}{(1-x^2)^2}\,dx = \int \frac{4\sin\theta}{\cos^4\theta}\cdot\cos\theta\,d\theta$$

$$= 4\int \frac{\sin\theta}{\cos^3\theta}\,d\theta \qquad \left\{\begin{array}{l} u = \cos\theta \\ du = -\sin\theta\,d\theta \\ -du = \sin\theta\,d\theta \end{array}\right\}$$

$$= -4\int \frac{1}{u^3}\,du$$

$$= 2u^{-2}+C$$

$$= \frac{2}{\cos^2\theta}+C$$

$$= \frac{2}{(\sqrt{1-x^2})^2}+C = \frac{2}{1-x^2}+C$$

(c) (2 points) partial fractions.

$$\frac{4x}{(1-x^2)^2} = \frac{4x}{[(1-x)(1+x)]^2} = \frac{4x}{(1-x)^2(1+x)^2} = \frac{4x}{(x-1)^2(x+1)^2}, \quad \text{so}$$

$$\frac{4x}{(x-1)^2(x+1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}$$

$$4x = A(x-1)(x+1)^2 + B(x+1)^2 + C(x+1)(x-1)^2 + D(x-1)^2$$

If $x=1$: $4 = 4B \Rightarrow B=1$

If $x=-1$: $-4 = 4D \Rightarrow D=-1$

Now, we have

$$4x = A(x-1)(x+1)^2 + (x+1)^2 + C(x+1)(x-1)^2 - (x-1)^2$$

$$4x = A(x^2-1)(x+1) + \cancel{x^2}+2x+\cancel{1} + C(x^2-1)(x-1) - \cancel{x^2}+2x-\cancel{1}$$

$$4x = A(x^3-x+x^2-1) + 4x + C(x^3-x-x^2+1)$$

$$4x = x^3(A+C) + x^2(A-C) + x(-A+4-C) - A + C$$

x^3: $0 = A+C$
x^2: $0 = A-C$
x: $4 = -A+4-C$
1: $0 = -A+C$

$\left.\right\}$ $A=-C$, $A=C$ $\Rightarrow$ $A=C=0$

Consequently:

$$\int \frac{4x}{(1-x^2)^2}\,dx = \int \frac{0}{x-1} + \frac{1}{(x-1)^2} + \frac{0}{x+1} + \frac{-1}{(x+1)^2}\,dx$$

$$= \int \frac{1}{(x-1)^2}\,dx - \int \frac{1}{(x+1)^2}\,dx$$

could do u-sub here

$$= \frac{-1}{x-1} + \frac{1}{x+1} + C$$

$$= \frac{-x-1+x-1}{(x-1)(x+1)} + C$$

$$= \frac{-2}{x^2-1} + C = \frac{2}{1-x^2} + C$$

2. (3 points) Evaluate $\int \frac{-x^3+5x^2-2x+7}{(x-1)(x^2+2)}\,dx$.

Degree on top: 3
degree on bottom: 3
> Same, so need to do long division.

$(x-1)(x^2+2) = x^3+2x-x^2-2$

$$x^3-x^2+2x-2 \;\overline{)\,-x^3+5x^2-2x+7}$$
quotient: -1
$$-(-x^3+x^2-2x+2)$$
$$4x^2+5$$

$\Rightarrow \frac{-x^3+5x^2-2x+7}{(x-1)(x^2+2)} = -1 + \frac{4x^2+5}{(x-1)(x^2+2)}$ ← degree on top smaller than deg. on bottom ✓

$$\frac{4x^2+5}{(x-1)(x^2+2)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+2}$$

$$4x^2+5 = A(x^2+2) + (Bx+C)(x-1)$$

$x=1$: $9 = A(3)+0 \Rightarrow A=3$

$x=0$: $5 = 2A + C(-1) \Rightarrow 5 = 6 - C \Rightarrow C=1$

$x=2$: $21 = 6A + (2B+C)\cdot 1 \Rightarrow 21 = 18 + 2B + 1 \Rightarrow 2 = 2B \Rightarrow B=2$

Consequently,

$$\int \frac{-x^3+5x^2-2x+7}{(x-1)(x^2+2)}\,dx = \int -1 + \frac{3}{x-1} + \frac{2x+1}{x^2+2}\,dx$$

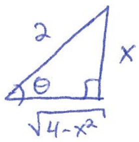
$$= -x + 3\ln|x-1| + \int \frac{2x}{x^2+2}\,dx + \int \frac{1}{x^2+2}\,dx$$

$\left\{ \begin{array}{l} u = x^2+2 \\ du = 2x\,dx \end{array} \right\}$

$$= -x + 3\ln|x-1| + \int \frac{1}{u}\,du + \frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right) + C$$

$$= \boxed{-x + 3\ln|x-1| + \ln|x^2+2| + \frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right) + C}$$

3. (2 points) Evaluate $\displaystyle\int_0^1 \frac{x^2}{(4-x^2)^{3/2}}\,dx$.



$\sin\theta = \frac{x}{2}$

$\boxed{x = 2\sin\theta}$

$\boxed{dx = 2\cos\theta\, d\theta}$

$\cos\theta = \frac{\sqrt{4-x^2}}{2}$

$\sqrt{4-x^2} = 2\cos\theta$

$\boxed{(4-x^2)^{3/2} = (2\cos\theta)^3}$

$$\int \frac{x^2}{(4-x^2)^{3/2}}\,dx = \int \frac{4\sin^2\theta}{8\cos^3\theta}\cdot 2\cos\theta\, d\theta$$

$$= \int \frac{\sin^2\theta}{\cos^2\theta}\, d\theta$$

$$= \int \frac{1-\cos^2\theta}{\cos^2\theta}\, d\theta$$

$$= \int \frac{+1}{\cos^2\theta} + -1\, d\theta$$

$$= \int +\sec^2\theta\, d\theta + -\int d\theta$$

$$= +\tan\theta - \theta + C$$

use triangle, $\sin\theta = x/2$, so $\theta = \arcsin(x/2)$

$$= \frac{+x}{\sqrt{4-x^2}} - \arcsin(x/2) + C$$

Now,

$$\int_0^1 \frac{x^2}{(4-x^2)^{3/2}}\,dx = \left[\frac{x}{\sqrt{4-x^2}} - \arcsin(x/2)\right]_0^1$$

$$= \left[\frac{1}{\sqrt{3}} - \arcsin(1/2)\right] - \left[0 - \arcsin(0)\right]$$

$$= \left(\frac{1}{\sqrt{3}} - \frac{\pi}{6}\right) - (0-0)$$

$$\boxed{= \frac{1}{\sqrt{3}} - \frac{\pi}{6}}$$